Why are collateral levels so extreme?

Agostino Capponi* W. Allen Cheng†

Abstract

We develop a game-theoretic model of centralized clearing to analyze a clearinghouse's choice of transaction fee and collateral requirements. The clearinghouse's requirements affect not only the size and riskiness of her participating client base, but also the transaction fees charged to clients by her clearing member. We show that empirically observed extreme collateral levels (compared to aggregate fees) can be explained as the equilibrium arising from strategic interactions between profit maximizing agents.

We analytically characterize the equilibrium fee-to-collateral ratio and find that it depends on the relative riskiness of the contract and the relative costliness of operating a client clearing business (both relative to the depth of clients' private trading benefits.) We observe that the clearinghouse tends to impose a very high collateral requirement when the contract is very risky, so that participating clients are mostly speculators as opposed to fundamental value traders, and impose lower requirements when the operational cost of client clearing is high, so to incentivize the member's participation.

1 Introduction

Clearinghouses use collateral, essentially a security that pays off only when there is a default, to protect themselves from counterparty risk. In comparison, the transaction fees that clearinghouses charge are guaranteed upfront payments in all states of nature, and are their main source of revenue. Since the default probability of each individual client is usually small, it might be tempting to conclude that the clearinghouse should be indifferent between increasing collateral levels by a large amount and increasing fees by a small amount.

However, this stands in contrast with empirical evidence. High collateral levels (relative to fees) consistently prevail in many derivatives markets, while low collateral levels are common among primary asset markets. For example, for the year 2014, a conservative estimate of Intercontinental Exchange's (ICE) collateral holdings pledged for cleared CDS trades was around 20 billion USD, while the company's revenue from CDS transactions totaled 161 million USD. Assuming an average five-year maturity of the CDS trades, this gives a fee-to-collateral ratio of around $\frac{161\times 5}{20{,}000} \approx 4\%$. This indicates that ICE

*Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027, USA, ac3827@columbia.edu

†Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027, USA, wc2232@columbia.edu

preferred collateral a lot more than fees. On the other hand, equity exchange-associated clearinghouses often charge almost negligible collateral compared to aggregate brokerage fees.

Intuitively, a clearinghouse desires high levels of collateral either if she deems the client to have high probability of default, or if she is very risk averse, i.e. she places significant emotional weight on the states of nature where a default occurs. Since a clearinghouse could decide to not clear contracts when clients are deemed likely to default, the more plausible explanation seems to be that high collateral clearinghouses are very risk averse.

This paper shows that this needs not be the case. The empirically observed extreme fee-to-collateral ratios can be explained as an equilibrium phenomenon arising from the strategic interactions between agents. We design a game where the risk-neutral profit maximizing clearinghouse clears client trades submitted via a clearing member bank. She sets her requirements (transaction fee and collateral) for cleared trades which directly affect not only the size and riskiness of the clearinghouse's participating client base, but also the transaction fees charged to clients by her clearing member. Clients have the option to default, and may do so when the contract value moves against them. The bank may choose to leave the clearinghouse if the client clearing business is not profitable. We show that the resulting equilibrium can only involve very low or very high collateral levels, depending on the riskiness of the traded contract and the benefits that clients can capture from trading.

The clearinghouse's requirements obviously reduce the clients' profit from trading: the transaction fee directly lowers the clients' overall surplus from trading, and collateral lowers the value of the client's default option. Since participating clients are those who can realize sufficient benefits from trading, increasing these requirements reduces the clearinghouse's participating client base. In addition, we show that the profit of the clearing member bank decrease when requirements are higher, so that the clearinghouse may lose profitable client business if the bank chooses to not become a clearing member.

The requirements also act as a screening device. When they are high, participating clients have smaller incentives to default either because they can capture large benefits from maintaining the trade or because they would lose large amounts of collateral if they default. This means that the fraction of participating clients who default, given a fixed realization of the contract value, is thus lower. The need for using such screening devices comes from the fact that the bank and clearinghouse either cannot or choose to not discriminate between clients. In practice, while clearinghouses often have access to the identities of the clients submitting trades, there are many clients for the clearinghouse to keep track of every one of them. Indeed, transaction fees and collateral are usually charged based on some portfolio specific rule rather than tailored to client specific characteristics.[1] In particular, firm specific characteristics such as credit quality

[1]ICE Clear Credit charges every clearing member with the same CDS portfolio the same amount of initial margin. Every client currently pays the transaction fee of $6 per million notional cleared for CDS index contracts. For more detail see ICE Clear Credit's schedule of fees and online documentation of margining rules.

and asset size often do not factor into the fee and collateral calculations.

Our model economy consists of three groups of risk-neutral agents: the clearinghouse, a potential clearing member bank, and a continuum of clients who may trade a single, mandatorily cleared, contract. The economy has two periods, *ex-ante* and *ex-post* the realization of the contract value. Prior to the realization, agents decide whether or not to participate in the clearing process. The clearinghouse sets her fee and collateral requirements. The bank then sets his own transaction fee and participates if the business is deemed profitable; that is, if his revenue can cover his operational costs for participation. High fee and collateral requirements both disincentivize the clients from trading. After the realization, clients default strategically, resulting in losses which are borne by the clearinghouse.[2] Client's trades are motivated by their private benefits and the value of their default option. Non-defaulting clients receive (heterogenous) private benefits when they carry out cleared trades. In (subgame perfect) equilibrium all agents choose strategies that maximize their profits, taking into account the reaction of others. We classify all possible equilibria of the model.

We find that when the operational cost of becoming a clearing member is low, so that the bank's individual rationality constraint is non-binding, the prevailing equilibrium is either one involving infinite collateral (when the contract is very risky) or one that involves zero collateral (when the contract is not risky). We find that it is the *relative* riskiness of the contract, measured by the volatility of the contract value over the depth of the private benefits to be realized, that determines which equilibrium prevails. This means that infinite collateral equilibria prevails when many clients trade for speculative purposes (when the default option value is high) and zero collateral equilibrium prevails when clients trade more for fundamental value (private benefits are large).

When the operational cost of becoming a clearing member is high, so that the bank's individual rationality constraint is binding, the clearinghouse can opt to set reduced requirements to increase the bank's profit, and incentivize him to participate as a clearing member. In this case the extreme collateral phenomenon still prevails, as long as it is individually rational for the clearinghouse to set up the clearing channel. The fact that the bank's individual rationality constraint may be binding can be observed from the recent exit of many clearing members. For instance, in May 2014, the Royal Bank of Scotland announced the wind down of its clearing business due to increasing operational costs. This was followed by State Street, BNY Mellon, and more recently, Nomura; each shutting down part or all of their OTC clearing business. This "constrained" equilibrium has certain welfare implications, as we demonstrate the possible existence of Pareto improvements in the model economy when this equilibrium prevails.

Our baseline model considers the case where collateral comes at no cost. The resulting equilibria can involve infinite collateral levels which, while serving well our purposes of explanation and demonstrating the intuition, is far from realistic. We consider also the case where collateral comes at a small but positive cost. We show that our qualitative results from the baseline model is naturally inherited while quantitatively we can

[2]We are implicitly assuming that the clearinghouse is acting as a true *central counterparty*. All counterparty risk and collateral management duties are borne by the clearinghouse.

calibrate the model to match empirically observed fee and collateral levels.

To the best of our knowledge, our paper is the first theoretical study to explain collateral levels in a centralized clearing setting. Previous work in the context of collateralized trading often assume that either margining rules are exogenously given (Garleanu and Pedersen (2011)), or that margins are set following some mix of the expected shortfall, VaR, and maximum shortfall measures (Anderson and Jõeveer (2014), Duffie et al. (2015)). Johannes and Sundaresan (2007) and Capponi (2013) assume that (variation) margin payments on the valuation of swaps are set exactly to track mark to market value changes. In contrast, our work looks for a micro-founded collateral setting rule via solving a profit maximization problem. Our modeling approach is closely related to those of Stiglitz and Weiss (1981) in the modeling of strategic defaults, and Holmström and Tirole (1997) in the inclusion of private benefits.

On the other hand, the determination of optimal levels of collateral has been extensively analyzed in the corporate finance literature. For instance, Stiglitz and Weiss (1981) and Besanko and Thakor (1987) both find collateral as a useful screening device either through adverse selection or incentive effects, and that collateral levels arise from profit-maximizing Nash equilibria. Geanakoplos (1997) analyzes (general) collateral equilibria for the case when assets are used to collateralize security trades, implicitly assuming the "smallness" of each agent. The centralized clearing setting is quite different, however. First, all agents trade publicly available contracts and are exposed to the same market risks, rather than bringing independent individual borrower risk to the table (Diamond (1984)), thus there is less asymmetric information about the risks clients are taking on. Second, the direction of future exposure is uncertain as either counterparty could be out of the money in the future. Third, collateral posting is not between clients, but is often unilateral from the clients to the clearinghouse. While initial margins is posted by clients to the clearinghouse, the clearinghouse usually does not post initial margins to the clients (Pirrong (2011)). Fourth, the clearinghouse often has significant market power.[3]

The rest of the paper is organized as follows. Section 2 explains the set up of the baseline model. Section 3 solves for and characterizes all possible equilibria. Section 4discusses welfare implications of the model. Section 5 analyzes the extended model where collateral comes at a small but nonzero cost. Section 6 concludes. All proofs are delegated to the appendix.

2 The model

Our model consists of three groups of risk neutral agents: the clearinghouse (CH), a potential clearing member bank (CM), and a continuum of clients trading a single, mandatorily cleared, derivative contract. There are two periods, separated by the realization of the contract value. Agents choose their participation in the trading/clearing process ex-ante; ex-post, clients default strategically and the clearinghouse bears the

[3] For instance, while ICE, CME, and LCH all clear CDS, the bulk of CDS are still cleared through ICE both in the US and Europe.

losses.[4] The clearinghouse has a large endowment of equity and does not default on her obligations. The bank only serves as an intermediary in the model so he does not default.

Before the contract value is realized, the clearinghouse can set her collateral C and fee δ_c requirements per contract cleared. The bank is the prime broker of a continuum of clients with unit mass, whose "private benefit parameters" B are described by the distribution F, i.e. $F(t)$ is the fraction of clients whose private benefits that do not exceed t. In view of C and δ_c, the bank can set his fee per contract cleared δ_b. After δ_c, C and δ_b have been declared, each client, characterized by his private benefit parameter B, has the choice between trading long one contract (L), trading short one contract (S), and not-trading (NT). These actions provide private benefit B, $-B$, and 0, respectively. Since each individual client is small, he cannot become a clearing member himself and cannot afford to trade more than one contract.[5]

Client choices are submitted to the bank. After receiving the trade orders, the bank has a choice between joining (J) and not joining (NJ) as a clearing member. The bank incurs an operational cost G for joining.[6] The bank's payoff is given by:

$$\text{Bank's Payoff} = \delta_b \times \text{mass of clients who trade} - G. \tag{2.1}$$

If the bank were to become a clearing member, a clearing channel is set up. Clients who trade pay the total fee of $\delta := \delta_b + \delta_c$ where δ_c goes to the clearinghouse and δ_b to the bank, before the contract value is realized. They also post collateral to the clearinghouse.

After this is done, the contract value is realized with value $\varepsilon \sim H$. The distributions F and H are assumed to be Laplace with parameters $(0, \lambda)$ and $(0, \gamma)$, respectively. That is, the cumulative distribution functions are

$$F(t) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda t}, t \geq 0 \\ \frac{1}{2}e^{\lambda t}, t < 0 \end{cases},$$

$$H(t) = \begin{cases} 1 - \frac{1}{2}e^{-\gamma t}, t \geq 0 \\ \frac{1}{2}e^{\gamma t}, t < 0 \end{cases}.$$

We use f and h to denote the associated density functions. Notice that while H is a probability distribution describing the random realization of a random variable ε, F is not; it is used to describe the distribution of deterministic private benefits among a unit mass of clients.

Before proceeding further, we discuss the economic interpretation of the parameters γ and λ. The mean absolute deviation of a Laplace $(0, \gamma)$ distribution is $\frac{1}{\gamma}$. γ serves as

[4]Since there are only two periods, it suffices to consider only initial margins and not variation margin posting.

[5]In practice, clients usually are asset management funds or money market funds and are very small compared to clearing members, who are usually large broker-dealers (Pirrong (2011)).

[6]As a clearing member, the bank needs to meet certain capital requirements, contribute to a default fund, set up operational channels for client clearing, and, in extreme circumstances, bear large losses of the clearinghouse.

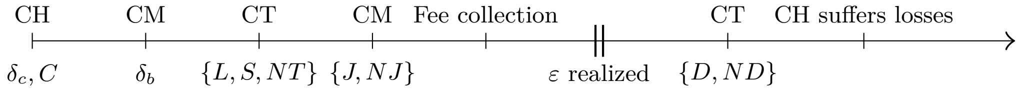


Figure 1: The extensive form game timeline

a measure of the *volatility* of the random contract value, with the contract value being more volatile when γ is small. It thus represents the extent to which clients trade due to speculation, generating surplus primarily from the default option. λ, on the other hand, serves as a measure of the *depth* of deteministic private benefits to be realized from financial trading. When λ is small, there are many clients with large private benefits. It thus represents the extent to which clients trade due to fundamental value, generating surplus primarily from capturing private benefits.

After the contract value is realized, clients default strategically. If a long client choose to not default (ND), he receives his private benefit B and the contract realization ε; if he chooses to default (D), he does not receive the private benefit and loses his collateral C. The short case is analogous: he either receives $-B$ and $-\varepsilon$ or loses C. They default whenever it is more profitable to do so.

Last, the clearinghouse collects payments from clients who are out of the money and is obligated to pay the clients who are in the money. When out-of-the-money clients default, the clearinghouse experiences a shortfall in payments and must make up for the difference using her own equity thereby incurring a loss. The clearinghouse's payoff is thus

$$\delta_c \times \text{client trades} - \text{loss per contract} \times \text{clients who trade and default.} \tag{2.2}$$

A graphical illustration of the timeline of the game is given in Figure 1.

3 Equilibrium collateral levels

In this section we solve for the subgame perfect Nash equilibria using backward induction. Throughout the paper a tilde sign denotes an implicitly defined function version of the variable, now as a function of the parameters of an underlying optimization problem or defining equation.

We first solve for clients' choice of defaults, assuming that the clearing channel has been set up. Since clients default whenever it is more profitable, the long client's payoff function is:[7]

$$\max(B - \delta + \varepsilon, -\delta - C). \tag{3.1}$$

The short client's payoff function is

$$\max(-B - \delta - \varepsilon, -\delta - C) \tag{3.2}$$

[7] In Section 5 and the majority of the proofs, we consider the generalized case where collateral comes at a cost $\alpha \geq 0$. In that case the long client's payoff is given by $\max(B - \delta - \alpha C + \varepsilon, -\delta - (1 + \alpha)C)$.

Notice that clients default only when the market moves against them. In particular, if $\varepsilon < 0$, all long buyers with insufficient private benefit $B + \varepsilon \leq -C$ will default; if $\varepsilon > 0$, all short buyers with insufficient (negative) private benefit $-B - \varepsilon \leq -C$ will default.

Next, the bank joins (J) as a clearing member if the client clearing business is profitable:

$$\delta_b \times \text{mass of clients who trade} - G \geq 0.$$

Clients trade when their ex-ante expected payoff is positive. Our first theorem shows that there is a unique private benefit threshold governing the trading decisions of the clients.

Theorem 1. *Fix $\delta, C \geq 0$, then there exists a unique trading threshold $\tilde{B} = \tilde{B}(\delta, C)$ such that a client wants to trade long if $B \geq \tilde{B}$, and wants to trade short if $B \leq -\tilde{B}$.*

A straightforward calculation (included in the appendix) shows that $\tilde{B}$ is the solution to

$$\delta = \tilde{B} + \int_{\tilde{B}+C}^{\infty} (1 - H(x))\, dx.$$

The above expression indicates that this "trading threshold" does not depend on the distribution of private benefits F. Indeed, after δ and C are declared, each client needs only evaluate the profitability of his own potential trade, disregarding the trading actions of his fellow clients. From Theorem 1 we see immediately that clearinghouse requirements create a "no-trade" region.

The clearinghouse may want the clients with private benefits $\tilde{B} > B > -\tilde{B}$ to trade. However, the requirements are such that this would not be profitable for these clients. In our model, the clearinghouse and bank only have information about the *distribution* of client private benefits and cannot distinguish between clients before the trades are submitted. Therefore they cannot incentivize more clients to trade by offering them reduced fees or lowered collateral levels. This phenomenon is illustrated in figure 2. We remark that since F is symmetric, the number of positions traded by each bank's clients net to zero (the market always clears).

Notice that a client with private benefits B such that $\tilde{B} \leq B \leq -\tilde{B}$ would want to be both long and short.[8] However, multiple contracts are usually governed by a *master agreement* that dictates that when a client defaults on one position, he must default on all positions governed by the agreement (Hull (2012)). Thus, in the case where $\tilde{B} \leq B \leq -\tilde{B}$, the client's payoff from trading both long and short a contract is

$$\max(B - \delta + \varepsilon - B - \delta - \varepsilon, -2\delta - 2C) = -2\delta - 2C.$$

Hence, he would never choose such a position when fees or collateral are positive. In this case the client chooses the option that gives higher expected payoff: long if $B > 0$ and short if $B < 0$. Thus the level of private benefits beyond which clients will trade

[8] Notice that this case exists if and only if $\tilde{B} \leq 0$.

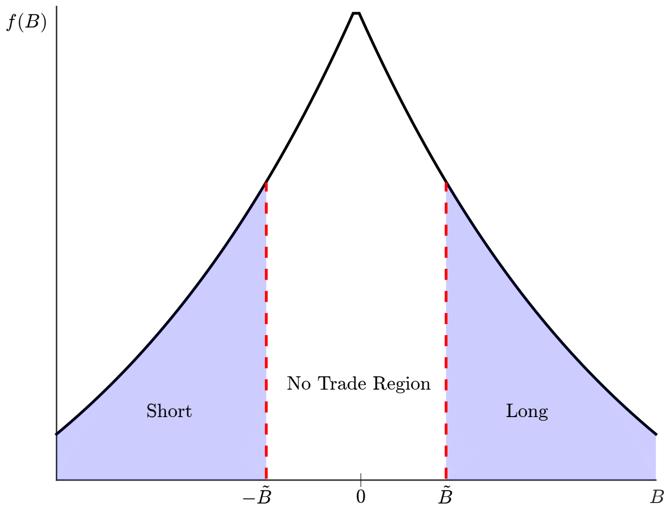


Figure 2: Each client trades depending on his level of private benefits. Only clients with high (low) benefits trade long (short).

long is bounded below by zero.[9] It will thus be convenient to define the *effective trading threshold* $\bar{B}(\delta, C) = \max(\tilde{B}(\delta, C), 0)$.

We can now express the bank's payoff function given in Eq. (2.1) as

$$R(\delta_b; \delta_c, C, G) := 2\delta_b(1 - F(\bar{B})) - G, \tag{3.3}$$

where $1 - F(\bar{B})$ is the fraction of clients who are long the contract.

The clearinghouse's payoff function is

$$X(\delta_c, C) := 2\delta_c(1 - F(\bar{B})) + (\varepsilon + C)(F(-C - \varepsilon) - F(\bar{B}))^+ + (-\varepsilon + C)(F(-C + \varepsilon) - F(\bar{B}))^+.$$

The first term corresponds to the clearinghouse's income from transaction fees. The second term is the aggregate loss from long clients who default: the loss per default $-\varepsilon - C$ multiplied by the mass of clients who traded long ex-ante and defaulted ex-post $(F(-C-\varepsilon)-F(\bar{B}))^+$. The last term is the aggregate loss from short clients who default. The case of long clients defaulting on a negative realization of ε is illustrated in Figure 3. When a low contract value ε_L is realized all clients with private benefits between $\bar{B}$ and $-C - \varepsilon$ have traded long the contract and choose to default, creating losses to the clearinghouse. When a high value ε_H is realized, there are no defaults, since the clients who would have defaulted did not trade in the first place.

The clearinghouse's expected payoff can be explicitly computed and is given by

$$E[X(\delta_c, C)] = \delta_c e^{-\lambda \bar{B}} - \frac{\lambda}{2(\lambda + \gamma)} e^{-\lambda \bar{B} - \gamma(\bar{B} + C)} \left(\frac{1}{\gamma} + \frac{1}{\lambda + \gamma} + \bar{B} \right). \tag{3.4}$$

We can now define our subgame perfect Nash equilibria.

[9] As we will see later, this effect of the master agreement induces a discontinuity in the rate at which the payoff of the bank changes with respect to the clearing requirements.

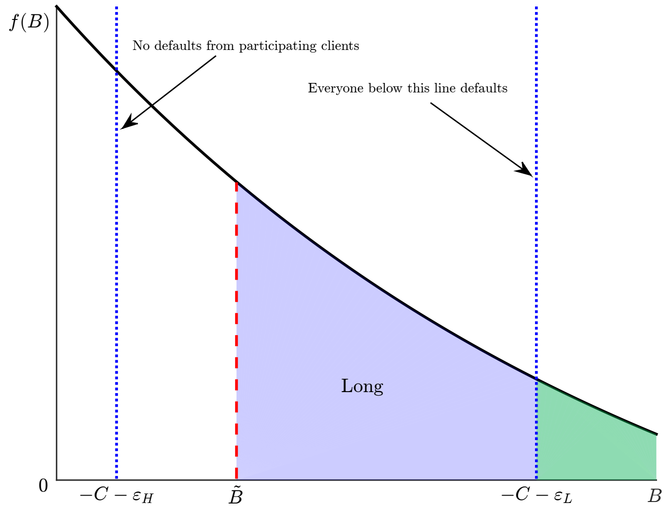


Figure 3: After the contract value is realized, clients with insufficient benefits default. When a low value ε_L is realized, clients who trade and default create losses for the clearinghouse. When a high value ε_H is realized, there are no defaults.

Definition 1. *An equilibrium is a triple* $(\tilde{\delta}_c, \tilde{C}, \tilde{\delta}_b(\cdot,\cdot))$ *such that*

$$(\tilde{\delta}_c, \tilde{C}) \in \underset{(\delta_c, C)\in R_+^2}{\operatorname{argmax}} \; E[X(\delta_c, C)]; \tag{3.5}$$

$$\tilde{\delta}_b(\delta_c, C) \in \underset{\delta_b \in R_+}{\operatorname{argmax}} \, R(\delta_b; \delta_c, C, G) \textit{ for all} \, (\delta_c, C) \in R_+^2; \tag{3.6}$$

$$R(\tilde{\delta}_b; \tilde{\delta}_c, \tilde{C}, G) \geq 0 \tag{3.7}$$

$$E[X(\tilde{\delta}_c, \tilde{C})] \geq 0. \tag{3.8}$$

Notice that we only focus on the clearinghouse and the bank's choice in equilibrium to reduce clutter. In equilibrium, the bank, in response to δ_c and C, chooses a fee level $\tilde{\delta}_b(\delta_c, C)$ that maximize his expected payoff. The clearinghouse takes this into account and sets $\delta_c = \tilde{\delta}_c$ and $C = \tilde{C}$ to maximize her profit. The last two constraints are the individual rationality (IR) constraints of the bank and the clearinghouse, respectively. We assume both the bank and the clearinghouse have a reservation value of 0: The bank only joins when his revenue covers his costs, and the clearinghouse will only initiate the business if her expected profit is nonnegative.[10]

We next show that the bank always has a unique choice of fees that maximizes his payoff:

Theorem 2. *Fix* $\delta_c, C \geq 0$*, then* $R(\delta_b; \delta_c, C, G)$ *has a unique local maximum at* $\tilde{\delta}_b = \tilde{\delta}_b(\delta_c, C)$*. In addition,* $\tilde{\delta}_b(\delta_c, C)$ *is continuous.*

[10]Although not setting up the clearing channel is not an explicit choice of strategy our model set up, the clearinghouse can effectively "kill the market" by choosing $\delta_c = \infty$.

Theorem 2 indicates there will not be a sudden "jump" in bank fees when the clearinghouse changes her requirements. The unambiguity of the bank's choice will play an important role in the analysis of the clearinghouse's action.

Interestingly, although $\tilde{\delta}_b(\delta_c, C)$ is continuous, depending on δ_c and C, the bank's response to increases in the clearinghouse's requirements fees may be increasing his fees (augmenting fees) or decreasing his fees (complementing fees). It will be convenient to define the "regime switching" function:

$$\xi(\delta_c, C) := 1 + \lambda\delta_c - \frac{1}{2}e^{-\gamma C}\left(1 + \frac{\lambda}{\gamma}\right) \tag{3.9}$$

Our next result gives the precise conditions identifying the two regimes.

Theorem 3. *The following statements hold:*

1. *(Augmenting fees.) If $\xi \geq 0$, $\tilde{\delta}_b$ is given as the unique solution to*

$$\gamma(\delta_c + C) + \log 2 - 1 = -(\lambda + \gamma)\delta_b - \log(1 - \lambda\delta_b), \tag{3.10}$$

   greater than or equal to $\frac{\gamma}{\lambda+\gamma}\frac{1}{\lambda}$. In this case $\frac{\partial\tilde{\delta}_b}{\partial\delta_c} = \frac{\partial\tilde{\delta}_b}{\partial C} \geq 0$, and $\tilde{B} \geq 0$.

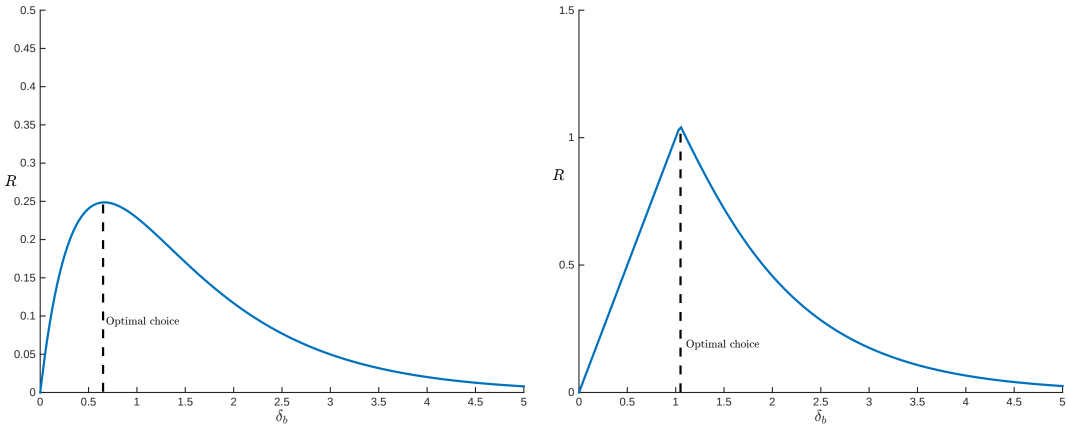
2. *(Complementing fees.) If $\xi < 0$,*

$$\tilde{\delta}_b = \frac{1}{2\gamma}e^{-\gamma C} - \delta_c. \tag{3.11}$$

   In this case $\frac{\partial\tilde{\delta}_b}{\partial\delta_c} < 0$, $\frac{\partial\tilde{\delta}_b}{\partial C} < 0$, and $\tilde{B} = 0$.

We refer to the first regime as the bank imposing *augmenting fees*. In this regime, when the clearinghouse increases requirements, the participating clients base shrinks; in response, the bank increases his fee, which further decreases the participating client base. However, the gain from increased fee outweighs the loss in the mass of participating clients. We refer to the second regime as the bank imposing *complementing fees*. In this regime, the entire client base is participating; when the clearinghouse increases requirements, the participating clients base again shrinks. Since the clearinghouse's requirements are low, the bank decreases his fee so that again the full client base is participating. In this case, the loss from the decreased fee is outweighed by the gain in the mass of participating clients. Some examples of the revenue functions (income as a function of the bank's fee) the banks face is given in Figure 4.

Additionally, Theorem 3 shows that the clearinghouse's strategy space is separated into two distinct regions. Which regime the bank adapts to depends on her choice of clearing requirements. Figure 5 illustrates this phenomenon. Notice that the complementing fee regime is feasible only when $\frac{\gamma}{\lambda} < 1$, i.e. when the contract is sufficiently risky.



(a) Augmenting fees. $(\delta_c, C) = (2, 1)$ (b) Complementing fees.$(\delta_c, C) = (1, 1)$

Figure 4: Revenue as a function of the bank's fees for $\gamma = 0.2$ and $\lambda = 1$. The bank's choice of fees is always unique.

After characterizing the bank's response, we can solve for the clearinghouse's action. For future purposes, it will be convenient to work with the normalized quantities:

$$\begin{aligned}
&\tilde{u} := \lambda\tilde{\delta}_b, \\
&v := \gamma\delta_c \\
&w := \gamma C, \\
&\theta := \frac{\gamma}{\lambda} \\
&\tilde{G} := \lambda G \\
&\text{Normalized bank's profit } := \lambda R(\delta_b; \delta_c, C, G), \\
&\text{Normalized clearinghouse's expected profit } := \gamma E[X(\delta_c, C)].
\end{aligned} \tag{3.12}$$

From this point on when we refer to profits, we refer to normalized profits unless specified otherwise.

When the clearinghouse sets her requirements, she must take into account the two possible regimes that can result from her actions and the bank's IR constraint. The geometry of the constraint is given by the following theorem:

Theorem 4. *The graph* $L := \{(\delta_c, C) | R(\tilde{\delta}_b(\delta_c, C); \delta_c, C, G) = 0\}$ *is a decreasing, convex,* $\mathcal{C}^1$ *curve. Increasing* G *shifts* L *downwards. In addition,* L *crosses the curve* $\xi = 0$ *at most once.*

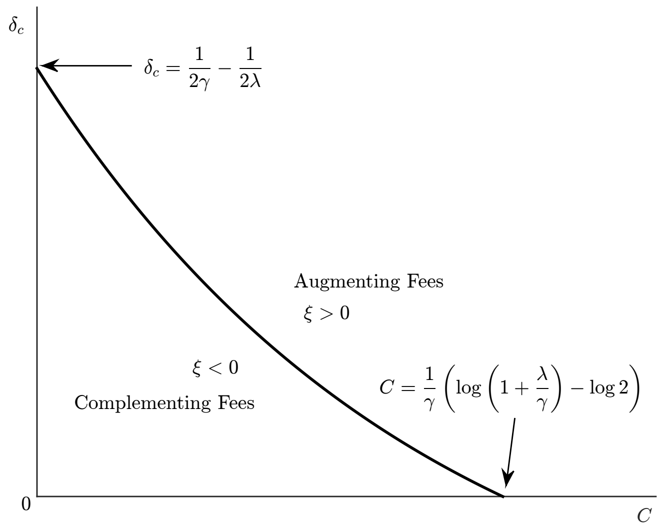


Figure 5: The clearinghouse's strategy space is separated into two sections. Depending on her choice of requirements, the bank reacts in different regimes. The complementing regime is feasible only when $\frac{\gamma}{\lambda} < 1$.

This geometry is demonstrated in Figure 6. As G increases, the feasible region in the $\delta_c - C$ plane is reduced.

We are now positioned to solve for the clearinghouse's equilibrium action.

Theorem 5. *The clearinghouses equilibrium action is one of five types.*

(i) (Unconstrained infinite collateral equilibrium) $(\tilde{u}, \tilde{v}, \tilde{w}) = (1, \theta, \infty)$. *The clearinghouse's (normalized, expected) profit is* e^{-2}. *The bank's (normalized) profit is* $e^{-2} - \tilde{G}$. *This can occur only when* $\tilde{G} \leq e^{-2}$.

(ii) (Constrained infinite collateral equilibrium) $(\tilde{u}, \tilde{v}, \tilde{w}) = (1, -\theta(\log \tilde{G} + 1), \infty)$. *The clearinghouse's profit is* $-\theta\tilde{G}(\log \tilde{G} + 1)$. *The bank's profit is* 0. *This can occur only when* $e^{-1} \geq \tilde{G} \geq e^{-2}$.

(iii) (Unconstrained zero collateral equilibrium) $\tilde{u}$ *is the unique solution to the equation*

$$-\log 2 + 1 - \log(1-u) + \frac{\theta^2}{u} - (1+\theta)^2 = 0$$

which is larger than $\frac{-\theta^2+\theta\sqrt{\theta^2+4}}{2}$. $(\tilde{v}, \tilde{w}) = \left(\frac{-\theta^2}{\tilde{u}} + (1+\theta)^2 - (1+\theta)\tilde{u}, 0\right)$. *The clearinghouse's profit is* $\frac{\theta\tilde{u}^2+(\theta^4+2\theta^3+2\theta^2)\tilde{u}-\theta^4-\theta^3}{(1+\theta)^2\tilde{u}}(2(1-\tilde{u}))^{\frac{1}{\theta}}$. *The bank's profit is* $\tilde{u}(2(1-\tilde{u}))^{\frac{1}{\theta}} - \tilde{G}$.

(iv) (Constrained zero collateral equilibrium) $\tilde{u}$ *is the unique solution to the equation*

$$\tilde{u}(2(1-\tilde{u}))^{\frac{1}{\theta}} = \tilde{G}$$

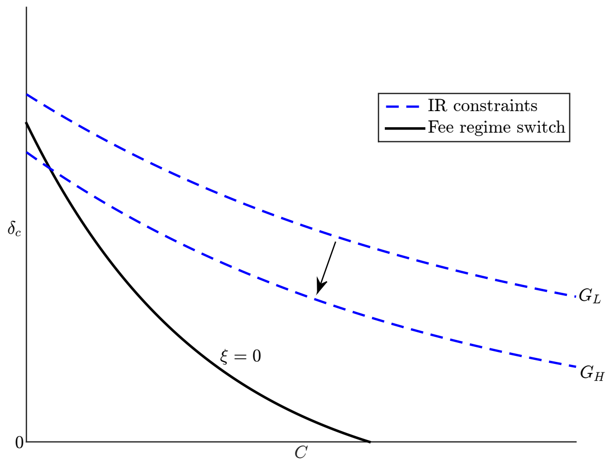


Figure 6: Bank's IR constraint. The bank's individual rationality constraint limits the feasible region of the clearinghouse's fee/collateral choices. Higher operational costs ($G_H > G_L$) reduce the feasible region.

which is larger than $\frac{\theta}{1+\theta}$. $(\tilde{v}, \tilde{w}) = (1 - \log 2 - (1+\theta)\tilde{u} - \log(1-\tilde{u}), 0)$. *The clearinghouse's profit is* $\frac{\tilde{G}}{\tilde{u}}\left(\frac{-(2+\theta)(\theta+(1+\theta+\theta^2)\tilde{u})}{(1+\theta)^2} + \frac{(\theta+\tilde{u})(\log 2 - \log(1-\tilde{u}))}{1+\theta}\right)$. *The bank's profit is* 0.

(v) (No clearing) *The clearinghouse does not set up the clearing channel. This can occur only when* $\tilde{G} \geq e^{-1}$.

In equilibria where the clearing channel is set up, the bank must be imposing augmenting fees. Which equilibrium prevails depends only on the value of θ *and* $\tilde{G}$.

Theorem 5 shows that the empirically observed low fee-to-collateral ratio can be explained as a result of risk neutral agents' strategic actions. What is most compelling is that observable equilibria either involves very high levels of collateral or very low levels of collateral, and which equilibrium prevails depends only on the *relative* riskiness of the contract and the operational cost of joining.

Consider first the case where $\tilde{G}$ is very low so that the banks' IR constraint is not-binding. In this case the prevailing equilibrium is either the unconstrained infinite collateral equilibrium or the unconstrained zero collateral equilibrium. We can easily demonstrate numerically that there exists situations in which either equilibrium is achieved. Table 1 gives some numerical values of the expected payoff function when $\lambda = 1$.

By plotting the optimized expected payoff as a function of γ, fixing $\lambda = 1$, we see an even stronger phenomenon. As illustrated in Figure 7, there is a critical value γ^* beyond which the clearinghouse chooses zero collateral, and below which the clearinghouse chooses infinite collateral. When γ is small, the contract value is very volatile

$\gamma \| (\delta_c, C, E[X(\delta_c, C)])$	Infinite Collateral	Zero Collateral
0.2	$(1, \infty, .1353)^*$	$(2.9493, 0, .1119)$
0.59	$(1, \infty, .1353)$	$(1.3261, 0, .1356)^*$
1	$(1, \infty, .1353)$	$(1.0681, 0, .1380)^*$
∞	$(1, \infty, .1353)^*$	$(1, 0, .1353)^*$

Table 1: Expected payoffs at possible extrema for $\lambda = 1$. Asterisks describe which equilibrium is chosen by the clearinghouse.

and the expected loss of the clearinghouse from client defaults is high. The clearinghouse thus chooses to eliminate defaults by setting high levels of collateral. When γ is large, the contract value is not as volatile and the expected loss of the clearinghouse from client defaults is low. The clearinghouse thus chooses to incentivize more clients to trade by setting collateral at very low levels. As γ approaches infinity, the contract value converges to zero. In this case the decisions of clients become independent of the collateral requirements, so that the zero collateral equilibrium and the infinite collateral equilibrium coincide and give the same optimized value.

The phenomenon observed in Figure 7 is quite general. By Theorem 5 we know that which equilibrium prevails when the bank's IR constraint is not binding depends only on the *relative* riskiness (relative to the depth of private benefits) of the contract value, θ. Since from Figure 7 the threshold $\gamma* \approx 0.58$, we know that $\theta^* = \gamma^*/1$ also serves as a threshold for all pairs (γ, λ): if $\theta > \theta^*$, the clearinghouse chooses zero collateral (infinite fee-to-collateral ratio), and infinite collateral (zero fee-to-collateral ratio) otherwise.

Next we consider general case when the cost of joining, G, is high, so that the bank's IR constraint may be binding. Due to the algebraic complexity of the expressions in Theorem 5, we numerically solve for the clearinghouse's expected profit for when she demands infinite or zero collateral, subject to the bank's IR constraint being satisfied. We record the equilibrium chosen for various values of θ and $\tilde{G}$ and present the results in Figure 8. Again, Theorem 5 implies that the phenomenon observed in Figure 8 is general for all triples (γ, λ, G).

We remark that all five equilibria outlined in Theorem 5 are observed in Figure 8. In addition, we observe even stronger phoenomena: (i) infinite collateral equilibria prevail when the contract is risky (θ is low), (ii)infinite collateral equilibria prevail when the bank's normalized operational cost is high, and that (iii) there exists situations where there is no equilibrium when the operational cost is too high. Indeed, when $\tilde{G}$ is too high, the clearinghouse must set requirements very low for the bank to receive enough revenue from his client clearing business; such requirements, however, mean the clearinghouse is taking losses, and it is therefore not individually rational for her to set up the clearing business. No clearing equilibria exist in this case.

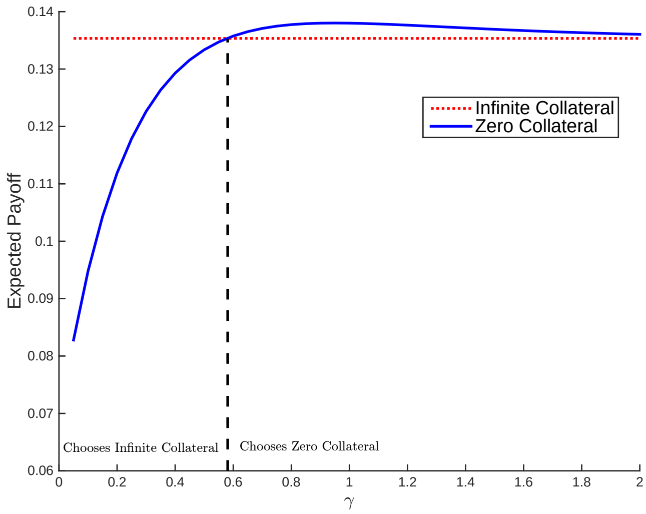


Figure 7: Maximum clearinghouse expected payoff as a function of γ for $\lambda = 1$. The clearinghouses chooses infinite collateral is below the threshold value $\gamma*$. Beyond this threshold value, the clearinghouse chooses zero collateral.

4 Pareto improvements

One of the most intriguing phenomena we observe in Figure 8 is the "taking over" of zero collateral equilibria by infinite collateral equilibria when the operational costs become higher. In this section we discuss how a social planner can make Pareto improvements to the model economy by increasing the operational cost $\tilde{G}$ via a lump sum tax when the equilibrium is constrained. The primary benefit of such a tax is to lower the fees imposed by the clearinghouse and stimulate trading activity in the model economy. When there are more trades, the captured additional surplus more than compensates for the decreased revenue.

Theorem 6. *Suppose the prevailing equilibrium is a constrained infinite collateral equilibrium. Assume that if the operational cost increases to $G + \tau$ a constrained infinite collateral equilibrium still prevails. Then social planner can improve social welfare by imposing a lump-sum tax τ on the bank, increasing the operational cost from G to $G+\tau$, and transferring the tax to the clearinghouse.*

When the bank's operational cost increases, the clearinghouse must decrease their fee's so that the bank still participates. The clearinghouse, receiving the lump sum transfer τ now has profit:

$$P(\tau) := \tau - (G + \tau)(\log \lambda(G + \tau) + 1).$$

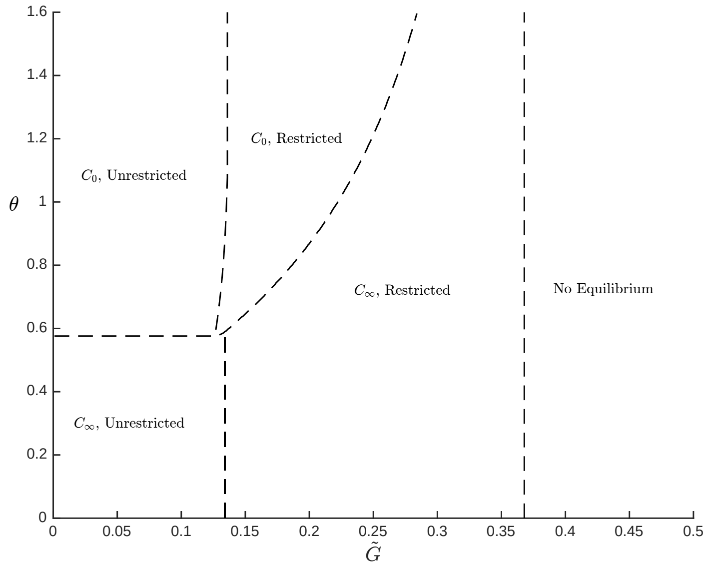


Figure 8: Prevailing equilibrium for various θ and $\tilde{G}$. Both high riskiness and high operational costs push the clearinghouse towards demanding infinite collateral (C_∞). There is no equilibrium when $\tilde{G}$ is too high.

Since the prevailing equilibrium is an infinite collateral equilibrium, we must have $\lambda(G+\tau) < e^{-1}$. This implies

$$\frac{\partial P}{\partial \tau} = -(\log \lambda(G+\tau) + 1) > 0, \text{ for } \lambda(G+\tau) < e^{-1}.$$

Thus, while the fees per client traded decrease, this is more than offset by the increased participation client base and lump sum transfer. The clearinghouse is better off. The bank is kept at zero profit and is indifferent. Since in any infinite collateral equilibrium bank's set $\tilde{u} = 1$, more clients trade and old clients pay a lower fee and are better off.

5 Costly Collateral

So far we have focused our analysis on the case where collateral is freely available to all clients. While this provides us with clear intuition about the arising equilibria as presented in Figure 8, the infinite collateral equilibrium clearly does not arise in practice. In this section we consider the case where collateral comes at a cost of α per unit. We show that the qualitative results presented in Section 3 remain the same: there exists equilibria that parallel the notions of zero and infinite collateral equilibria, where equilibria involve collateral levels that are "large but finite." For this section we restrict our attention to the cases where the bank's IR constraint is non-binding.

The payoff function for the long client for costly collateral is:

$$\max(B - \delta + \varepsilon - \alpha C, -\delta - (1+\alpha)C). \tag{5.1}$$

Since our proofs in Appendix A take into account collateral costs, one can easily see that the following analogies of Theorems 1–4 hold true:

1′. There is a unique trading threshold $\tilde{B}$ for $(\delta, C) \in R_+^2$. The market always clears.

2′. The banks always has a unique choice of fees $\tilde{\delta}_b$ which is continuous in the clearinghouse's requirements.

3′. Define $\xi_\alpha := \xi + \lambda\alpha C$. The bank imposes augmenting fees when $\xi_\alpha \geq 0$, given as the the unique solution to

$$\gamma(\delta_c + (1+\alpha)C) + \log 2 - 1 = -(\lambda+\gamma)\delta_b - \log(1 - \lambda\delta_b)$$

greater than or equal to $\frac{\gamma}{\gamma+\lambda}\frac{1}{\lambda}$. Otherwise $\delta_b = \frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C$.

4′. The bank's IR constraint is decreasing and convex.

In what follows we will restrict our attention to the case when the bank's IR constraint is non-binding (e.g. $G = 0$). Our first result characterizes the possible equilibrium for costly collateral

Theorem 7. *Let* $\left(\tilde{\delta}_c, \tilde{C}, \tilde{\delta}_b(\cdot,\cdot)\right) =: \left(\frac{\tilde{v}}{\gamma}, \frac{\tilde{w}}{\gamma}, \frac{\tilde{u}}{\lambda}\right)$ *be an equilibrium. Assume* $G = 0$ *and* $\alpha \geq 0$. *Then one of the following hold:*

(i) $\tilde{u}$ *solves*

$$\eta(\tilde{u}; \theta, \alpha) = 0,$$

where

$$\eta(u) := -2\theta - \theta^2 + \frac{\theta^2}{u} - \log(2 - 2\tilde{u}),$$

and

$$\tilde{v} = \frac{-\theta^2 + (1+\theta)^2\tilde{u} - (1+\theta)\tilde{u}^2}{\tilde{u}}.$$
$$\tilde{w} = 0.$$

(ii) $\tilde{u}$ *solves*

$$\kappa(\tilde{u}; \theta, \alpha) = 0,$$

where

$$\kappa(u; \theta, \alpha) := \alpha^2(1+\theta)^2 + \alpha(\theta^3 + 3\theta^2 + \theta + (1 + \theta - 2\theta^2 - \theta^3)\tilde{u}) - \theta(\theta + \theta^2 - \tilde{u})(\tilde{u} - 1) - \alpha(1+\theta)(\tilde{u} - 1)\log(2 - 2\tilde{u}).$$

and

$$\tilde{v} = \frac{\alpha - \theta^2(1+\alpha) + ((1+\theta)^2 + (-1+\theta+\theta^2)\alpha)\tilde{u} - (1+\theta)\tilde{u}^2 - \alpha(1-\tilde{u})\log(2-2\tilde{u})}{\alpha+\tilde{u}},$$

$$\tilde{w} = \frac{-(1+\theta)\tilde{u} - \log(2-2\tilde{u}) + 1 - v}{1+\alpha} = \frac{\theta^2 - 2\theta\tilde{u} - \theta^2\tilde{u} - \tilde{u}\log(2-2\tilde{u})}{\alpha+\tilde{u}}.$$

The first case in Theorem 7 shows that the zero collateral equilibrium is naturally inherited into the case of costly collateral. In fact, since zero collateral equilibria maximizes clearinghouse profit when the clearinghouse's requirements are restricted to $C = 0$, the clearinghouse's requirements are independent of α for zero collateral equilibria.

The second case parallels the case of unconstrained infinite collateral as given in Theorem 5. We perform a analysis of the equilibria arising when collateral costs are "local to zero" in Theorem 8, in which we demonstrate that collateral levels that prevail in equilibrium can be large and approach infinity as α go to zero.

Theorem 8. *For each* $\theta \in \left[0, \frac{-1+\sqrt{5}}{2}\right]$*, there exists* $\tilde{\alpha}(\theta)$ *such that for* $A := [0, \tilde{\alpha}(\theta)]$*, there exists a differentiable function* $u_\theta^\infty : A \to \left[\frac{\theta}{1+\theta}, 1\right]$ *such that* $u_\theta^\infty(\alpha)$ *solves* $\kappa(\tilde{u}; \theta, \alpha) = 0$*. In addition,*

1. $\lim_{\alpha->0} u_\theta^\infty(\alpha) = 1$,
2. $\frac{\partial u_\theta^+}{\partial \alpha} < 0$.
3. *The associated equilibrium fee* $\tilde{v} = \tilde{v}(\alpha, u_\theta^\infty(\alpha))$ *increases with* α*, while the equilibrium collateral level* $\tilde{w} = \tilde{w}(\alpha, u_\theta^\infty(\alpha))$ *decreases with* α*.*

In what follows we will refer to $u_\theta^\infty(\alpha)$ as the *infinite collateral path.* Theorem 8 implies that for small α and $\theta < 0.618$,[11] when α increases the clearinghouse's requirements change in such a way that the bank's normalized fees move along the infinite collateral path. This corresponds to decreases in the collateral and fee requirements. Intuitively, as collateral becomes costlier to come by, less clients are willing to trade. In view of this, the clearinghouse shifts away from collateral requirements and asks for more fees. This in turn reduces the bank's fees.

To see how general are the local results presented in Theorem 8, we numerically solve for the fees and (unnormalized) collateral levels arising along the infinite collateral path and present them in Figure 9 for $\gamma = 0.2$ and $\lambda = 1$.

Given γ and λ, one can easily calibrate our extended model to explain observed fee and collateral levels. For instance, keeping $\gamma = 0.2$ and $\lambda = 1$, a collateral cost of two to three basis points (corresponding to ratios of 3.9 and 4.25 percent, respectively) matches the empirically observed four percent fee-to-collateral ratio of ICE Clear Credit.

In Figure 10 we present collateral levels for various values of θ and α along the infinite collateral path. We observe that collateral levels generally decrease as α increases,

[11]Notice that from our numerical results in Figure 7, all values of θ where the unconstrained infinite collateral equilibrium prevails are under 0.6.

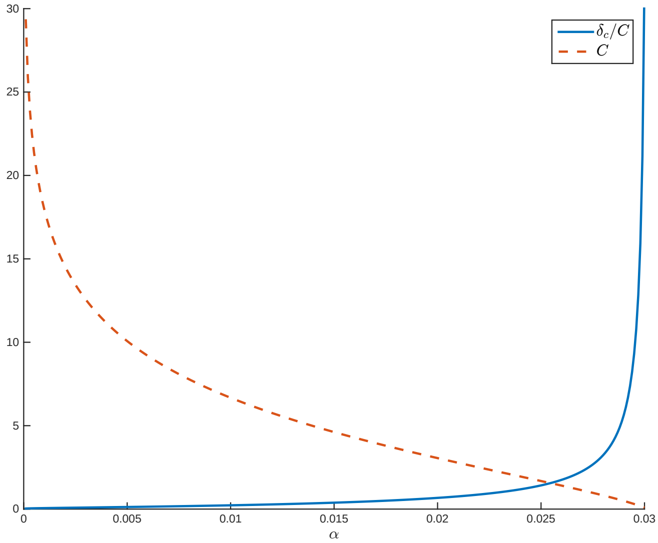


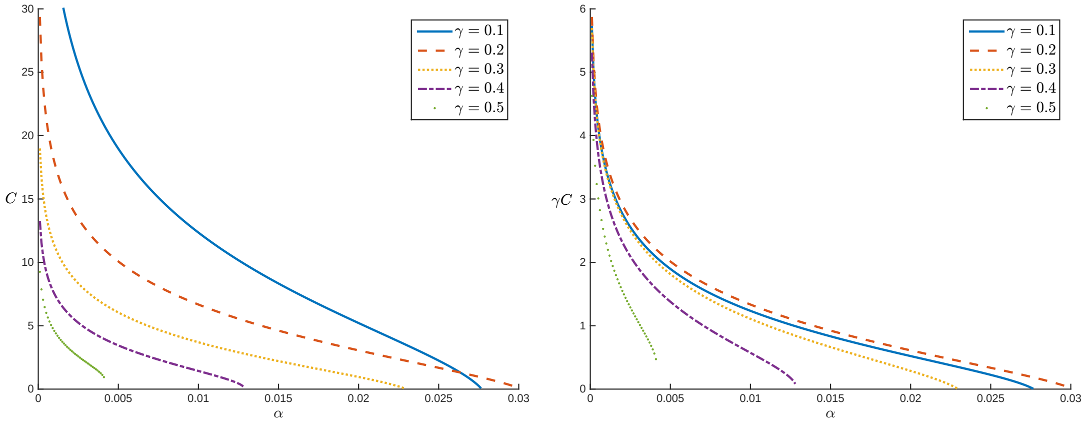
Figure 9: Fee to collateral ratios and collateral levels along the branch of solutions $u^\infty(\alpha)$ for $\gamma = 0.2$ and $\lambda = 1$. Collateral levels decrease with cost.

following the results of Theorem 8. At the same time, we observe another interesting phenomenon: for fixed α collateral levels are not a monotonic function of θ. Indeed, for certain levels of alpha, we see that collateral levels, both normalized and unnormalized, increase when we move from $\theta = 0.1$ to $\theta = 0.2$, but then decrease when we move further to $\theta = 0.3$.

We offer an explanation to this phenomenon as follows. There are two counteracting effects when θ increases. When the contract is riskier, the clearinghouse would like to charge more collateral to protect herself against default losses. At the same time, more clients are willing to trade since the value of the default option is higher. In the case of our baseline model when collateral comes at no cost, the clearinghouse can effectively lower the portion of "speculating" clients by setting collateral to infinity.

When collateral comes at a cost, clients essentially pay an upfront payment to creditors to obtain collateral. This creates a leak of value out of the the system since the upfront payment does not increase the clearinghouse's profit. When the contract is riskier, more clients are willing to borrow collateral and speculate on the high value of the default option. The lowered aggregate surplus actually lowers the clearinghouse's profit and thus she would rather lower her collateral requirement.

Our numerical results show that for certain parameter values, this second effect can dominate and result in the clearinghouse lowering collateral requirements when the contract becomes riskier.



(a) Collateral levels

(b) Normalized collateral levels

Figure 10: Collateral levels along the infinite collateral path for various γ and $\lambda = 1$. While collateral levels generally decrease with costs, they do not necessarily increase with riskiness.

6 Concluding remarks

Our model of centralized clearing is designed to explain the extreme collateral levels observed in the market. Our results indicate that fee and collateral choices of the clearinghouse affect her payoffs in two different ways: they influence the payoff per client cleared and the mass of clients who trade. We have analyzed the equilibrium that results from the profit maximization process of all agents in the model. We demonstrated the possible existence of Pareto improvements in equilibrium, and provided a calibratable framework with collateral costs. It should be noted that we are not asserting that the low fee-to-collateral ratios must result from strategic expected profit maximization, but rather propose it as a plausible alternative to assigning agents some (unobservable) high level of risk aversion and utility constraints.

Our choice of the Laplace distribution is due to the straightforward economic interpretation of its parameters and the analytical tractability. However, it is also desirable to assess the robustness of our results with respect to different distributional specifications of private benefits and contract value.

Our results can be easily extended to the case of multiple clearing members when the clearinghouse sets requirements separately for each clearing member's client base. A more interesting but difficult extension is to consider multiple clearing members where the clearinghouse is restricted to set uniform requirements for clients clearing via all member banks. In this case one must take into account all of the private benefit distributions and the relative sizes of the potential client bases that can be cleared through the clearing member. We expect that when one clearing member has many more clients than others, the clearinghouse would choose requirements close to the case of one member treated in this paper.

One important part of the default waterfall we do not address is the clearinghouse's choice of guaranty fund requirements. In our framework, there is no need for guaranty funds since the clearing member does not default. One future line of research would be to analyze how clients would endogenize potential clearing member defaults and how this affects the equilibrium collateral levels.

A Proofs

In this section we restate and prove the generalized versions of the theorems in section 3, taking into account the cost of collateral (see section 5) . The results presented in section 3 can be obtained by setting $\alpha = 0$.

Theorem 1a. *Fix $\delta, C, \alpha \geq 0$, then there exists a unique trading threshold $\tilde{B} = \tilde{B}(\delta, C; \alpha)$ such that a client wants to trade long if $B \geq \tilde{B}$, and wants to trade short if $B \leq -\tilde{B}$.*

Proof of Theorem 1a. We start with the long case. Assume $\tilde{B}$ is a solution to

$$\delta + \alpha C = E[(B+\varepsilon)1_{\varepsilon > -B-C} - C1_{\varepsilon \leq -B-C}] \tag{A.1}$$

That is, $\tilde{B}$ is a level of private benefit at which expected profit for the client is zero. Set

$$\phi_1(B) := E[(B+\varepsilon)1_{\varepsilon > -B-C} - C1_{\varepsilon \leq -B-C}],$$

we see that

$$\begin{aligned}\lim_{B\to\infty} \phi_1(B) &= \lim_{B\to\infty} E[B1_{\varepsilon > -B-C}] + E[\varepsilon 1_{\varepsilon > -B-C}] - E[C1_{\varepsilon \leq -B-C}] \\ &= \infty + 0 - C = \infty.\end{aligned}$$

by using the fact that $E[\varepsilon] = 0$ and applying the monotone convergence theorem. In addition,

$$\begin{aligned}\lim_{B\to-\infty} \phi_1(B) &= \lim_{B\to-\infty} -(-B-C)E[1_{\varepsilon > -B-C}] + E[\varepsilon 1_{\varepsilon > -B-C}] - C \\ &= -C,\end{aligned}$$

where we have used the fact that $E|\varepsilon| < \infty$ implies that $\lim_{x\to\infty} xP(\varepsilon > x) = 0$. Thus, if $\delta, C, \alpha \geq 0$, there must exist $B = \tilde{B}$ such that Eq. (A.1) is satisfied.

Next, notice that $\phi_1(B)$ is a strictly increasing function of B, since

$$\phi_1(B) = \int_{-B-C}^{\infty} (B+x)\, dH(x) + \int_{-\infty}^{-B-C} -C\, dH(x),$$

$$\phi_1'(B) = 1 - H(-B - C) > 0.$$

So the solution must be unique. Since H is symmetric, the short case follows by a symmetry argument. □

Theorem 2a. *Fix $\delta_c, C, \alpha \geq 0$, then $R(\delta_b; \delta_c, C, G)$ has a unique local maximum at $\tilde{\delta}_b = \tilde{\delta}_b(\delta_c, C; \alpha)$. In addition, $\tilde{\delta}_b(\delta_c, C; \alpha)$ is continuous in δ_c and C.*

Before the proof of Theorem 2a, we introduce two auxiliary propositions:

Proposition 1. *Let $\delta, C, \alpha \geq 0$. Then $\tilde{B}$ solves*

$$\delta + \alpha C = \tilde{B} + \frac{1}{2\gamma} e^{-\gamma(B+C)}. \tag{A.2}$$

In addition $\tilde{B} \geq 0$ if and only if

$$\delta + \alpha C \geq \frac{1}{2\gamma} e^{-\gamma C}.$$

Proof of Proposition. By definition

$$\begin{aligned}
\delta + \alpha C &= \int_{-\tilde{B}-C}^{\infty} (\tilde{B} + x)\, dH(x) + \int_{-\infty}^{-\tilde{B}-C} -C\, dH(x) \\
&= \tilde{B} - (\tilde{B} + C) H(-\tilde{B} - C) + \int_{-\tilde{B}-C}^{0} x\, dH(x) + \int_{0}^{\infty} x\, dH(x) \\
&= \tilde{B} - (\tilde{B} + C) H(-\tilde{B} - C) + xH(x)\Big|_{-\tilde{B}-C}^{0} - \int_{-\tilde{B}-C}^{0} H(x)\, dx + \int_{0}^{\infty} (1 - H(x))\, dx \\
&= \tilde{B} - \int_{0}^{\tilde{B}+C} (1 - H(y))\, dy + \int_{0}^{\infty} (1 - H(x))\, dx \\
&= \tilde{B} + \int_{\tilde{B}+C}^{\infty} (1 - H(x))\, dx
\end{aligned}$$

Here we used the layer cake representation of expectation to derive the third equality and the fact that H is a symmetric distribution to derive the fourth equality. Now plug $\tilde{B} = 0$ into Eq. (A.2), we have

$$\delta + \alpha C = \int_{C}^{\infty} (1 - H(x))\, dx = \frac{1}{2\gamma} e^{-\gamma C}.$$

Notice that

$$\frac{\partial}{\partial B} \left(B + \int_{B+C}^{\infty} (1 - H(x))\, dx \right) = H(B + C) > 0,$$

so $\tilde{B}(\delta, C; \alpha) \geq 0$ if and only if $\delta \geq \frac{1}{2\gamma} e^{-\gamma C}$. □

The next proposition follows immediately from differentiating Eq. (A.2):

Proposition 2.

$$\frac{\partial \tilde{B}}{\partial \delta} = \frac{1}{H(\tilde{B}+C)} > 0$$
$$\frac{\partial \tilde{B}}{\partial C} = \frac{1+\alpha}{H(\tilde{B}+C)} - 1 \geq 0$$

Proof of Theorem 2a. We first show the existence of a maximizer. Define the bank's revenue function as

$$\phi_2(\delta_b) := \delta_b(1 - F(\bar{B}(\delta_b + \delta_c, C; \alpha))).$$

By Proposition 1, for large enough δ_b, we have $\bar{B}(\delta_b + \delta_c, C; \alpha) = \tilde{B}(\delta_b + \delta_c, C; \alpha)$. By Proposition 2, $\frac{\partial \tilde{B}}{\partial \delta} > 1$ since H is a distribution function. In addition, $\bar{B}$ is continuous since $\tilde{B}$ is continuous. Thus, it follows that

$$\lim_{\delta_b \to \infty} \tilde{B}(\delta_b + \delta_c, C; \alpha) = \infty \tag{A.3}$$
$$\lim_{\delta_b \to \infty} \frac{\delta_b}{\tilde{B}(\delta_b + \delta_c, C; \alpha)} < \infty. \tag{A.4}$$

Since $\int_0^\infty t dF(t) = \frac{1}{2\lambda} < \infty$, it follows by dominated convergence:

$$\lim_{B \to \infty} B(1 - F(B)) = 0,$$

which implies,

$$\lim_{\delta_b \to \infty} \phi_2(\delta_b) = \lim_{\delta_b \to \infty} \frac{\delta_b}{\tilde{B}(\delta_b + \delta_c, C; \alpha)} \lim_{\tilde{B} \to \infty} \tilde{B}(1 - F(\tilde{B})) = 0.$$

Where we used Eq. (A.3) and (A.4) to derive the second limit. Since $\phi_2(\delta_b) \geq 0$ and $\phi_2(0) = \phi_2(\infty) = 0$, and ϕ_2 is continuous, there exists an interior maximizer of $\phi_2(\delta_b)$ on $(0, \infty)$.

Next we show uniqueness. Fix $\delta_c, C, \alpha \geq 0$. By Proposition 1, the bank's payoff function can be written as

$$2\delta_b(1 - F(\bar{B})) - G = \begin{cases} \delta_b e^{-\lambda \tilde{B}(\delta_b + \delta_c, C)} - G, & \delta_b + \delta_c + \alpha C \geq \frac{1}{2\gamma} e^{-\gamma C} \\ \delta_b - G, & \delta_b + \delta_c + \alpha C < \frac{1}{2\gamma} e^{-\gamma C} \end{cases} \tag{A.5}$$

Suppose $\tilde{\delta}_b \in \left\{ \delta_b \middle| \tilde{B}(\delta_b + \delta_c, C; \alpha) < 0 \right\} = \left\{ \delta_b \middle| \delta_b + \delta_c + \alpha C < \frac{1}{2\gamma} e^{-\gamma C} \right\}$ is a local maximizer of ϕ_2, then ϕ_2 can be always increased by choosing δ_b slightly larger than $\tilde{\delta}_b$, thus $\tilde{\delta}_b$ cannot be a local maximizer. Therefore, a local maximizer must be in the region $\{\delta_b | \tilde{B}(\delta_b + \delta_c, C; \alpha) \geq 0\}$. This implies that a local maximizer $\tilde{\delta}_b$ either solves the equation

$$0 = \tilde{B}(\delta_b + \delta_c, C; \alpha), \tag{A.6}$$

or is a critical point of the differentiable function

$$\phi_3(\delta_b) := \delta_b e^{-\lambda \tilde{B}(\delta_b+\delta_c, C;\alpha)}.$$

Obviously, if Eq. (A.6) has a nonnegative solution, it must be $\delta_b^0 := \frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C$, in this case the bank's payoff is exactly $\delta_b^0 - G$ by Eq (A.5). Also by Eq (A.5), if a local maximizer is a critical point of ϕ_3, it must be larger than δ_b^0.

We now analyze the critical points of the function ϕ_3 strictly larger than δ_b^0. We will show there is at most one such critical point.

The first order condition is

$$0 = 1 - \frac{\lambda\delta_b}{1 - \frac{1}{2}e^{-\gamma(\tilde{B}(\delta_b+\delta_c, C;\alpha)+C)}} \tag{A.7}$$

By Eq. (1), we also have

$$\delta_b + \delta_c + \alpha C = \tilde{B} + \frac{1}{2\gamma}e^{-\gamma(\tilde{B}+C)} \tag{A.8}$$

Thus, combining Eq. (A.7) and (A.8) any critical point of $\phi_3(\delta_b)$ must satisfy

$$\gamma(\delta_c + (1+\alpha)C) + \log 2 - 1 = -\lambda\delta_b - \gamma\delta_b - \log(1-\lambda\delta_b) \tag{A.9}$$

Define

$$\phi_4(\delta_b) := -\lambda\delta_b - \gamma\delta_b - \log(1-\lambda\delta_b),$$

and

$$\phi_5(\gamma) := \gamma(\delta_c + C) + \log 2 - 1.$$

We see that

$$\begin{aligned}
&\phi_4(0) = 0 \\
&\lim_{\delta_b \to 1/\lambda} \phi_4(\delta_b) = \infty \\
&\phi_4''(\delta_b) = \frac{\lambda^2}{(1-\lambda\delta_b)^2} \geq 0, \text{ for } \delta_b \in (0, \lambda^{-1}).
\end{aligned}$$

Now, the unique minimum of $\phi_4(\delta_b)$ is given by the first order condition:

$$\begin{aligned}
\phi_4'(\delta_b) &= -\lambda - \gamma + \frac{\lambda}{1-\lambda\delta_b} = 0. \\
\delta_b^* &= \frac{\gamma}{\lambda(\lambda+\gamma)} < \frac{1}{\lambda} \qquad \text{(A.10)} \\
\phi_4(\delta_b^*) &= -\frac{\gamma}{\lambda} + \log\left(1 + \frac{\gamma}{\lambda}\right) < 0
\end{aligned}$$

The above analysis shows that on $\{\delta_b | 0 < \lambda\delta_b < 1\}$, $\phi_3(\delta_b)$ has exactly one critical point when $\phi_5(\gamma) > 0$ or $\phi_5(\gamma) = \phi_4(\delta_b^*)$ and exactly two critical points when $\phi_4(\delta_b^*) < \phi_5(\gamma) < 0$.

We will now show that when there are two critical points, the smaller critical point is always less than δ_b^0. Define

$$\begin{aligned}\phi_7 &:= \gamma(\delta_c + (1+\alpha)C),\\ u &:= \lambda\delta_b,\\ \theta &:= \frac{\gamma}{\lambda},\end{aligned}$$

then we can rewrite Eq. (A.9) as:

$$\phi_7 + \log 2 - 1 = -(1+\theta)u - \log(1-u). \tag{A.11}$$

Fixing $\phi_7 \geq 0$, we see that

$$\min_{\{(\delta_c,C)\in R_+^2 \mid \gamma(\delta_c+(1+\alpha)C)=\phi_7\}} \delta_b^0 = \frac{1}{\gamma}\left(\frac{1}{2} - \phi_7\right) =: \bar{m}(\phi_7)$$

$$\operatorname*{argmin}_{\{(\delta_c,C)\in R_+^2 \mid \gamma(\delta_c+(1+\alpha)C)=\phi_7\}} \delta_b^0 = \left(\frac{\phi_7}{\gamma}, 0\right)$$

It is clear that there exists, for some choice of $\lambda, \gamma, \delta_c, C$, two critical points of $\phi_3(x)$ greater than δ_b^0 if and only if for some $\phi_7 \in [0, 1-\log 2]$, the smaller solution to Eq.(A.11) is larger than $\lambda\bar{m}(\phi_7)$.

Denote the smaller branch of solutions to Eq. (A.11) as $u^-(\phi_7)$. It holds that,

$$\lambda\bar{m}(1-\log 2) = \frac{\log 2 - \frac{1}{2}}{\theta} > u^-(1-\log 2) = 0.$$

By Eq. (A.10), we see that $u^-(\phi_7) < \frac{\theta}{1+\theta}$. Since $-(1+\theta) + \frac{1}{1-u} > 0$ for all $u \in [0, \frac{\theta}{1+\theta})$ we have:

$$\frac{\partial}{\partial\phi_7}\left(\lambda\bar{m}(\phi_7) - u^-(\phi_7)\right) = -\frac{1}{\theta} - \frac{1}{-(1+\theta)+\frac{1}{1-u^-}} = -\frac{\frac{u^-}{1-u^-}}{\theta\left(-(1+\theta)+\frac{1}{1-u^-}\right)} < 0$$

Thus $\bar{m}(\phi_7) > u^-(\phi_7)$ for $Y \in [0, 1-\log 2]$. Thus there is at most one critical point, $\delta_b^\dagger$, greater than δ_b^0.

Since the payoff function is differentiable everywhere but at δ_b^0, the function cannot be maximized simultaneously at $\delta_b^\dagger$ or δ_b^0, unless they are equal. Indeed, using the Mean Value Theorem we have that if $\delta_b^\dagger \neq \delta_b^0$, either $\phi(\delta_b^\dagger) > \phi(\delta_b^0)$ or $\phi(\delta_b^\dagger) < \phi(\delta_b^0)$. Again since there is no critical point in between, only one of them can be a local maximizer.This allows us to conclude there is a unique maximizer: the function is either maximized at $\delta_b^\dagger$ or δ_b^0.

The second statement of the theorem, i.e., the continuity of the function $\tilde{\delta}_b(\delta_c, C)$, follows from Berge's Maximum Theorem. □

Theorem 3a. *Define*

$$\xi_\alpha := 1 + \lambda\delta_c + \alpha\lambda C - \frac{1}{2}e^{-\gamma C}\left(1 + \frac{\lambda}{\gamma}\right). \tag{A.12}$$

Then the following statements hold:

1. *(Augmenting fees.) If $\xi_\alpha \geq 0$, $\tilde{\delta}_b$ is given as the unique solution to*

$$\gamma(\delta_c + (1+\alpha)C) + \log 2 - 1 = -(\lambda+\gamma)\delta_b - \log(1 - \lambda\delta_b),$$

greater than or equal to $\frac{\gamma}{\lambda+\gamma}\frac{1}{\lambda}$. In this case $\frac{\partial\tilde{\delta}_b}{\partial\delta_c} = \frac{1}{1+\alpha}\frac{\partial\tilde{\delta}_b}{\partial C} \geq 0$, and $\tilde{B} \geq 0$.

2. *(Complementing fees.) If $\xi_\alpha < 0$,*

$$\tilde{\delta}_b = \frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C.$$

In this case $\frac{\partial\tilde{\delta}_b}{\partial\delta_c} < 0$, $\frac{\partial\tilde{\delta}_b}{\partial C} < 0$, and $\tilde{B} = 0$.

Proof of Theorem 3a. Define $\delta_b^0 := \frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C$.

Suppose $\delta_b^0 \leq 0$. By Proposition 1 we have $\tilde{B}(\delta_b + \delta_c, C; \alpha) > 0$ for all $\delta_b \geq 0$, thus $\tilde{\delta}_b$ must be the larger solution to Eq. (3.10) by Eq. (A.9). Notice that $\xi_\alpha < 0$ implies that $\delta_b^0 \geq 0$, so $\xi_\alpha \geq 0$ when $\delta_b^0 \leq 0$.

Now suppose $\delta_b^0 > 0$. we see the right derivative of the bank's payoff function Eq. (3.3) at δ_b^0 is given by

$$\lim_{\delta_b \to \delta_b^0} 1 - \frac{\lambda\delta_b}{1 - \frac{1}{2}e^{-\gamma(\tilde{B}(\delta_b+\delta_c,C;\alpha)+C)}} = 1 - \frac{\lambda(\frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C)}{1 - \frac{1}{2}e^{-\gamma C}}$$
$$\propto 1 - \frac{1}{2}e^{-\gamma C} - \lambda(\frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C) = 1 + \lambda\delta_c + \lambda\alpha C - \frac{1}{2}e^{-\gamma C}(1 + \frac{\lambda}{\gamma}).$$

Since the local maximum of the payoff function is unique by Theorem 2, $\tilde{\delta}_b$ must be the larger solution to Eq. (3.10) when the right derivative is positive, and equal to δ_b^0 when the right derivative is negative. The sign of the derivatives follow from direct differentiation. □

Before the proof of Theorem 4, we summarize the relations between the implicitly defined functions so far. All relations come from rearranging previous results or direct differentiation. Recall the definitions of normalized quantities from Eq. (3.12).

Corollary 1. *Define $K_\alpha := \{(\delta_c, C) | \delta_c \geq 0, C \geq 0, \xi_\alpha \geq 0\}$. Then in the interior of K_α, the following relations hold:*

$$1 - \tilde{u} = \frac{1}{2}e^{-\gamma\tilde{B} - w} \tag{A.13}$$

$$\theta\tilde{u} + v + \alpha w = \gamma\tilde{B} + \frac{1}{2}e^{-\gamma\tilde{B} - w} \tag{A.14}$$

$$\gamma\tilde{B} = (1+\theta)\tilde{u} + v - 1 + \alpha w \tag{A.15}$$

$$v + (1+\alpha)w + \log 2 - 1 = -(1+\theta)\tilde{u} - \log(1-\tilde{u}). \tag{A.16}$$

$$\gamma\tilde{B} = \frac{1+\theta}{1+\alpha}\tilde{u} + \frac{v}{1+\alpha} - \frac{1}{1+\alpha} - \frac{\alpha}{1+\alpha}\log(2-2\tilde{u}) \tag{A.17}$$

$$\frac{\partial\tilde{\delta}_b}{\partial\delta_c} = \frac{1}{1+\alpha}\frac{\partial\tilde{\delta}_b}{\partial C} = \frac{\theta}{-(1+\theta)+\frac{1}{1-\tilde{u}}}. \tag{A.18}$$

Moreover, $\tilde{u} \in [\frac{\theta}{1+\theta}, 1]$.

Theorem 4a. *Define the IR correspondences as*

$$\delta_c^L(C,G) := \{\delta_c | R(\tilde{\delta}_b(\delta_c, C; \alpha); \delta_c, C, G) = 0\},$$
$$C^L(\delta_c, G) := \{C | R(\tilde{\delta}_b(\delta_c, C; \alpha); \delta_c, C, G) = 0\}.$$

Then the following statements hold

1. *Fix* $C \geq 0$. *Then* $\delta_c^L(\cdot, G)$ *is a singleton for all* $G \geq 0$. *The function* $\delta_c^L(C, \cdot)$ *is decreasing.*

2. *Fix* $\delta_c \geq 0$. *Then* $C^L(\delta_c, \cdot)$ *is a singleton for all* $\delta_c \geq 0$. *The function* $C^L(\delta_c, \cdot)$ *is decreasing.*

3. *Fix* $G \geq 0$. *Then* $\delta_c^L(\cdot, G)$ *is a singleton for all* $C \geq 0$. *The function* $\delta_c^L(\cdot, G)$ *is* $\mathcal{C}^1$, *decreasing, and convex.*

4. *For any* $G \geq 0$, *the IR constraint crosses the regime switching line* $\xi = 0$ *line at most once.*

To prove Theorem 4a, we will make use of the following proposition:

Proposition 3. *Along* $L = \{(\delta_c, C) | R = 0\}$, *the following statements hold:*

1. *When the bank is imposing augmenting fees* $\frac{\partial\delta_c(C)}{\partial C} = \tilde{u} - 1 - \alpha < 0$.

2. *When the bank is imposing complementing fees* $\frac{\partial\delta_c(C)}{\partial C} = -\frac{1}{2}e^{-w} - \alpha < 0$.

Proof of Proposition. We first consider the case when banks are imposing augmenting fees, then by the Envelope Theorem, we can write

$$0 = \left(\frac{\partial\delta_c}{\partial C}\frac{\partial}{\partial\delta_c} + \frac{\partial}{\partial C}\right)\tilde{\delta}_b e^{-\lambda\tilde{B}(\tilde{\delta}_b+\delta_c, C;\alpha)},$$
$$\frac{\partial\delta_c}{\partial C} = H(\tilde{B}+C) - 1 - \alpha = \tilde{u} - 1 - \alpha.$$

Here we have used Eq. (A.13) from Corollary 1.

When the bank is imposing complementing fees, we have $G = \tilde{\delta}_b = \frac{1}{2\gamma}e^{-\gamma C} - \delta_c - \alpha C$. Thus

$$\frac{\partial\delta_c}{\partial C} = -\frac{1}{2}e^{-\gamma C} - \alpha.$$

□

Proof of Theorem 4a. Holding C fixed, we see that

$$\frac{\partial R(\tilde{\delta}_b;\delta_c,C,G)}{\partial G}+\frac{\delta_c(G)}{\partial G}\frac{\partial R(\tilde{\delta}_b;\delta_c,C,G)}{\partial \delta_c}=0,$$

By the Envelope Theorem and Proposition 3, we have

$$\frac{\partial \delta_c(G)}{\partial G}=\frac{1}{\tilde{u}-1}<0.$$

Thus for each C and G there can be only one δ_c^L satisfying the IR constraint. From Proposition 3 we can see that δ_c^L is decreasing. Analogous arguments show that $\delta_c^L(C,\cdot)$ and $C^L(G,\cdot)$ are all singleton-valued correspondences, decreasing in their arguments.

Further differentiation of the results in Proposition 3 gives

1. (Augmenting fees) $\frac{\partial^2 \delta_c^L(\cdot,G)}{\partial C^2}=\lambda\left(\frac{\partial \delta_b}{\partial \delta_c}\frac{\partial \delta_c}{\partial C}+\frac{\partial \delta_b}{\partial C}\right)=\lambda\frac{\partial \delta_b}{\partial \delta_c}\left(H(B+C)-1-\alpha+1+\alpha\right)>0$,
2. (Complementing fees) $\frac{\partial^2 \delta_c^L(\cdot,G)}{\partial C^2}=\frac{\gamma}{2}e^{-\gamma C}>0$.

$\delta_c^L(\cdot,G)$ is thus convex.

To show the "single crossing" property, observe that along the curve $\xi_\alpha=0$ we have

$$\frac{\partial \delta_c}{\partial C}\Big|_{\xi_\alpha=0}=-\alpha-\frac{1}{2}e^{-\gamma C}\left(1+\gamma\right)<-\alpha-\frac{1}{2}e^{-\gamma C}=\frac{\partial \delta_c}{\partial C}\Big|_{complem.}\leq-\alpha-\frac{1}{2}e^{-\gamma(\bar{B}+C)}=\frac{\partial \delta_c}{\partial C}\Big|_{aug.}.$$

Thus the curves can cross at most once if they cross at all.

For the $\mathcal{C}^1$ statement in Theorem 4, we see that to check for continuity of the derivative, we only need to check along the regime switching line $\xi_\alpha=0$ for $\alpha=0$, when augmenting fees and complementing fees agree. We have:

$$\lambda\delta_b-1=\frac{\lambda}{2\gamma}e^{-\gamma C}-(1+\lambda\delta_c)=-\frac{1}{2}e^{-\gamma C}.$$

Thus $\delta_c^L(\cdot,G)$ is $\mathcal{C}^1$ when $\alpha=0$. □

Before we prove Theorem 5, we present three useful propositions.

Proposition 4. *The first order condition* $\frac{\partial E[X]}{\partial \delta_c}=0$ *implies*

$$v=\frac{\alpha-\theta^2(1+\alpha)+((1+\theta)^2+(-1+\theta+\theta^2)\alpha)\tilde{u}-(1+\theta)\tilde{u}^2-\alpha(1-\tilde{u})\log(2-2\tilde{u})}{\alpha+\tilde{u}}. \tag{A.19}$$

This in conjunction with the first order condition $\frac{\partial E[X]}{\partial C}=0$ *implies*

$$v=\frac{\theta+(1+\theta+\alpha\theta)(\theta^2+\alpha(1+\theta)^2)}{(1+\theta)(\alpha(1+\theta)+\tilde{u}+\theta)}+\frac{(1+\theta)^2+\alpha(\theta^2-\theta-1)}{(1+\theta)(\alpha(1+\theta)+\tilde{u}+\theta)}\tilde{u}$$

$$-\frac{(1+\theta)}{\alpha(1+\theta)+\tilde{u}+\theta}\tilde{u}^2+\frac{\alpha(\tilde{u}-1)\log(2-2\tilde{u})}{\alpha(1+\theta)+\tilde{u}+\theta} \tag{A.20}$$

and that $\tilde{u}$ solves

$$0=\kappa(\tilde{u};\theta,\alpha), \tag{A.21}$$

where

$$\begin{aligned}\kappa(u;\theta,\alpha) &:= \alpha^2(1+\theta)^2+\alpha(\theta^3+3\theta^2+\theta+(1+\theta-2\theta^2-\theta^3)\tilde{u})\\ &-\theta(\theta+\theta^2-\tilde{u})(\tilde{u}-1)-\alpha(1+\theta)(\tilde{u}-1)\log(2-2\tilde{u}).\end{aligned}$$

Proof of Proposition 4. Define Λ as

$$\Lambda(\delta_c,C,B):=\delta_c e^{-\lambda B}-\frac{\lambda}{2(\lambda+\gamma)}e^{-\lambda B-\gamma(B+C)}\left(\frac{1}{\gamma}+\frac{1}{\lambda+\gamma}+B\right).$$

Then

$$\begin{aligned}\frac{\partial\Lambda}{\partial\delta_c}&=e^{-\lambda B}\\ \frac{\partial\Lambda}{\partial C}&=e^{-\lambda B}\left(e^{-\gamma(B+C)}\frac{\lambda\gamma}{2(\lambda+\gamma)}\left(B+\frac{\lambda+2\gamma}{\gamma(\lambda+\gamma)}\right)\right)\\ \frac{\partial\Lambda}{\partial B}&=e^{-\lambda B}\left(-\lambda\delta_c-\frac{\lambda}{2(\lambda+\gamma)}e^{-\gamma(B+C)}+\frac{\lambda}{2}e^{-\gamma(B+C)}\left(B+\frac{\lambda+2\gamma}{\gamma(\lambda+\gamma)}\right)\right)\end{aligned}$$

Evaluating each derivative at $B=\tilde{B}$, we obtain

$$e^{\lambda\tilde{B}}\frac{\partial\Lambda}{\partial\delta_c}=1. \tag{A.22}$$

$$e^{\lambda\tilde{B}}\frac{\partial\Lambda}{\partial C}=\frac{1-\tilde{u}}{1+\theta}\left(\frac{(1+\theta)\tilde{u}}{1+\alpha}+\frac{v}{1+\alpha}+\frac{\theta}{1+\theta}+\frac{\alpha}{1+\alpha}-\frac{\alpha}{1+\alpha}\log(2-2\tilde{u})\right) \tag{A.23}$$

$$e^{\lambda\tilde{B}}\frac{\partial\Lambda}{\partial B}=\frac{1}{\theta}\left(-v+(1-\tilde{u})(\gamma\tilde{B}+1)\right) \tag{A.24}$$

$$=\frac{1}{\theta}\left(\frac{(1+\theta)(1-\tilde{u})}{1+\alpha}\tilde{u}+\frac{-\alpha-\tilde{u}}{1+\alpha}v+\frac{(1-\tilde{u})\alpha}{1+\alpha}(1-\log(2-2\tilde{u}))\right) \tag{A.25}$$

$$1+\frac{\partial\tilde{\delta}_b}{\partial\delta_c}=\frac{\lambda^2\tilde{\delta}_b}{\lambda-(1-\tilde{u})(\lambda+\gamma)}=\frac{\tilde{u}}{1-(1-\tilde{u})(1+\theta)} \tag{A.26}$$

The first order condition with respect to δ_c is

$$0=\frac{\partial\Lambda}{\partial\delta_c}+\frac{\partial\Lambda}{\partial B}\frac{\partial B}{\partial\delta}\left(1+\frac{\partial\tilde{\delta}_b}{\partial\delta_c}\right) \tag{A.27}$$

After some algebra, this gives Eq. (A.19).

The first order condition with respect to C is

$$0 = \frac{\partial \Lambda}{\partial C} + \frac{\partial \Lambda}{\partial B}\left(\frac{\partial B}{\partial \delta}\frac{\partial \tilde{\delta}_b}{\partial C} + \frac{\partial B}{\partial C}\right) \tag{A.28}$$

$$= \frac{\partial \Lambda}{\partial C} + \frac{\partial \Lambda}{\partial B}(1+\alpha)\left(\frac{\partial B}{\partial \delta}\frac{\partial \tilde{\delta}_b}{\partial \delta_c} + (1+\alpha)\frac{\partial B}{\partial \delta} - 1\right) \tag{A.29}$$

$$= \frac{\partial \Lambda}{\partial C} - (1+\alpha)\frac{\partial \Lambda}{\partial \delta_c} - \frac{\partial \Lambda}{\partial B} \tag{A.30}$$

After some algebra, this gives Eq. (A.20).

Since at a critical point both first order conditions must hold, we have that the left hand sides of the two first order conditions are equal. Rearranging, we find that $\tilde{u}$ solves Eq. (A.21). □

Proposition 5. *Define $\phi_8(\delta_c) := E[X(\delta_c, 0)]$. Then there exists a unique δ_c^* at which ϕ_8 is maximized and $\xi_\alpha > 0$. This maximized value is positive. In addition, the bank's fees $\tilde{u} := \lambda\tilde{\delta}_b(\delta_c^*, 0)$ is the unique solution to $\eta(\tilde{u}) = 0$ greater than $\frac{\theta}{1+\theta}$, where*

$$\eta(u) := -2\theta - \theta^2 + \frac{\theta^2}{u} - \log(2 - 2\tilde{u}). \tag{A.31}$$

Proof of Proposition 5. We start with existence. Set $C = 0$. By Theorem 3, the bank imposes augmenting fees when

$$\xi_\alpha(\delta_c, 0) = 1 + \frac{1}{\theta}v - \frac{1}{2}\left(1 + \frac{1}{\theta}\right) \geq 0,$$

where we are using the definition of ξ_α given in Eq. (A.12). This is equivalent to

$$v \geq \frac{1-\theta}{2}.$$

We first consider the point $v = \frac{1-\theta}{2}$, i.e. the threshold above which the bank switches from imposing complementing to augmenting fees. Notice this is only relevant when $\theta < 1$. At this point, we have from Eq. (A.14), $\tilde{u} = \frac{1}{\theta}\left(\frac{1}{2} - \frac{1-\theta}{2}\right) = \frac{1}{2}$ and $\tilde{B} = 0$. The right derivative of ϕ_8 is given by:

$$e^{-\lambda\tilde{B}}\left(1 + \frac{1}{\theta}((1+\theta)(1-\tilde{u}) - v)\frac{\tilde{u}}{1-(1-\tilde{u})(1+\theta)}\right). \tag{A.32}$$

and is equal to

$$e^{-\lambda\tilde{B}}\left(1 + \frac{1}{\theta}((1+\theta)(1-\tilde{u}) - v)\frac{\tilde{u}}{1-(1-\tilde{u})(1+\theta)}\right)\Bigg|_{\tilde{u}=\frac{1}{2}, v=\frac{1-\theta}{2}} = 1 + \frac{1}{1-\theta} > 0.$$

at $v = \frac{1-\theta}{2}$. Thus the clearinghouse's profit is increasing at this point. Next, consider a point $v > \frac{1-\theta}{2}$, then by Eq. (A.16), we have

$$v + \log 2 - 1 = -(1+\theta)\tilde{u} - \log(1-\tilde{u}), \tag{A.33}$$

thus when v increases, $\tilde{u}$ approaches 1 monotonically. Thus there exists some $v^*(\theta)$ such that for $v > v^*(\theta)$ we have

$$\frac{\partial E\left[X(\delta_c, 0)\right]}{\partial \delta_c} < 0$$

by Eq. (A.32). Combined with the previous analysis this means that expected profit is maximized at a finite point where banks are strictly imposing augmenting fees. Since $E[X(\infty, C)] = 0$, this maximized value is positive.

The condition for optimality is then given by Eq. (A.19) and (A.33). Rearranging, this means $\tilde{u}$ solves $\eta(\tilde{u}) = 0$. Notice that this function always has exactly one zero on $u \in [\frac{\theta}{1+\theta}, 1)$.

$$\begin{aligned}
&\eta\left(\frac{\theta}{1+\theta}\right) = -\theta - \log 2 + \log(1+\theta) < 0 \\
&\eta(1^-) = \infty, \\
&\eta'(u) = 0 \text{ only at } u^* = \frac{\theta^2 + \theta\sqrt{\theta^2+4}}{2} \\
&\eta(u^*) = \frac{2\theta^2}{\theta\sqrt{\theta^2+4} - \theta^2} - \log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2+4}}{2}\right) - (\theta+1)^2 + 1 - \log 2 \\
&= \frac{-\theta^2 + \theta\sqrt{\theta^2+4}}{2} - \log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2+4}}{2}\right) - 2\theta - \log 2 < 0,
\end{aligned}$$

the last inequality follows from the following two inequalities

$$\theta \geq -\log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2+4}}{2}\right) \geq \frac{-\theta^2 + \theta\sqrt{\theta^2+4}}{2} \tag{A.34}$$

which holds true for all $\theta > 0$. To show the first inequality, we notice that $\theta = -\log\left(1 - \frac{-\theta^2+\theta\sqrt{\theta^2+4}}{2}\right)$ at $\theta = 0$. In addition, a cumbersome but straightforward computation shows that

$$\frac{\partial}{\partial \theta}\left(\theta + \log\left(1 - \frac{-\theta^2 + \theta\sqrt{\theta^2+4}}{2}\right)\right) = 1 - \frac{2}{4+\theta^2} \geq 0$$

Thus the first inequality in (A.34) holds. The second inequality follows from basic calculus since $-\log(1-x) \geq x$ for all $x \in [0, 1)$. Since ϕ_8 has at exactly one critical point, and we have shown that local maxima do not occur on the boundary, it must be the global maximum. □

Proposition 6. *Fix $\theta > 0, \alpha \geq 0$, then no local maxima of $E[X(\delta_c, C)]$ can occur on (i) $\delta_c = 0$, (ii) $C = 0$, and (iii) the regime switching line $\xi_\alpha = 0$.*

Proof of Proposition 6. Assume the contrary. Along $\xi_\alpha = 0$, we can consider the maximization problem:

$$\max_{\delta_c, C} E[X(\delta_c, C)]$$
$$\text{subject to } \xi_\alpha = 1 + \lambda\delta_c + \lambda\alpha C - \frac{1}{2}e^{-\gamma C}(1 + \frac{\lambda}{\gamma}) = 0.$$

Notice that when $\xi = 0$, $\bar{B} = 0$ by Theorem 3 and Proposition 1. Introducing the Lagrange multiplier μ, we obtain that the following must hold at such an optimum:

$$0 = 1 + \mu\lambda$$
$$0 = \frac{\gamma^2}{2}e^{-\gamma C}\left(\frac{1}{\gamma} - \frac{1}{\lambda+\gamma}\right)\left(\frac{1}{\gamma} + \frac{1}{\lambda+\gamma}\right) + \mu\left(\alpha\lambda + \frac{\gamma}{2}e^{-\gamma C}(1 + \frac{\lambda}{\gamma})\right).$$

Rearranging

$$\alpha = \frac{1}{2}e^{-\gamma C}\frac{-\theta^3 - 3\theta^2 - \theta}{(1+\theta)^3}.$$

Since this system of equations has no solution, a local maxima cannot occur along the boundary $\xi = 0$ unless $\delta_c = 0$ or $C = 0$. Yet Proposition 5 shows that there cannot be a local maximum at the point where $\xi_\alpha = 0$ and $C = 0$ intersect. Proposition 5 also shows that there is a choice of requirements where the clearinghouse is making positive expected profit, and thus it would never choose $\delta_c = 0$ since in that case expected profit is nonpositive. □

For Theorem 5 we consider only the case when $\alpha = 0$.

Proof of Theorem 5. Fix a triple $(\gamma, \theta, G) \in R^3$. We first define a pair of normalized requirements (v, w) to be: a zero collateral equilibrium if $w = 0$; an interior collateral equilibrium if $0 < w < \infty$; and an infinite collateral equilibrium if $w = \infty$. The equilibrium is constrained if the bank's profit at that pair of requirements is zero, and is unconstrained otherwise. Since the bank is can be imposing augmenting fees or complementing fees, this leads to 12 cases to be considered. The proof is broken down into four steps.

[Step 1.] We first rule out all unconstrained equilibria with complementing fees (3 cases). By Theorem 3, when the bank is imposing complementing fees we must have $\xi \leq 0$. We have $\tilde{B} = 0$ and the clearinghouse's expected payoff function Eq. (3.4) is therefore given as

$$E[X(\delta_c, C)] = \delta_c - \frac{\gamma}{2}e^{-\gamma C}\left(\frac{1}{\gamma} - \frac{1}{\lambda+\gamma}\right)\left(\frac{1}{\gamma} + \frac{1}{\lambda+\gamma}\right)$$

Thus, when $\xi < 0$, the clearinghouse can increase profit by increasing δ_c and C, and thus in unconstrained equilibria with complementing fees they must choose δ_c, C such that

$\xi = 0$. which is ruled out by Proposition 6. This rules out all unconstrained equilibria with strictly complementing fees.

[Step 2.] Next we show the possible existence of unconstrained infinite and zero collateral equilibria with strictly augmenting fees (cases *(i)* and *(iii)*), while ruling out unconstrained interior collateral equilibria with augmenting fees (3 cases). We search for all possible maxima over the space $K_0 := \{(\delta_c, C)|\delta_c \geq 0, C \geq 0, \xi \geq 0\}$. The payoff function is obviously continuous, and is continuously differentiable on $K_0 \backslash \{\xi(\delta_c, C) = 0\}$, where ξ is defined in Eq. (3.9). Notice that maxima along the boundary of K_0 are ruled out by Proposition 6, thus we need only consider maxima in the interior of K_0.

Using the expressions given by Proposition 4, we see that at critical point:

$$0 = -\frac{\theta^2(\tilde{u}-1)(\tilde{u}-\theta^2-\theta)}{\tilde{u}(1+\theta)(\theta+\tilde{u})}$$

The above equation implies that critical points can occur only at $(\tilde{u}_\infty, v_\infty) := (1, \theta)$ and $(\tilde{u}_{int}, v_{int}) := (\theta+\theta^2, 1-\theta^3-\frac{\theta^3}{1+\theta})$. $(\tilde{u}, v) = (1, \theta)$ directly implies $C = \infty$ by Eq. (A.16). These results, along with Eq. (A.16), give what we refer to as the (unconstrained) infinite and interior collateral equilibrium, respectively.

When $C = \infty$, the expression for $\tilde{B}$ simplifies to $\tilde{B} = \delta$ and $E\left[X(\delta_c, \infty)\right] = \delta_c e^{-\lambda(\delta_b+\delta_c)}$. In addition, $\tilde{u} = 1$. This fully characterizes the infinite collateral equilibrium in case *(i)*.

We can then rule out the interior collateral equilibrium with $\tilde{u}_{int} = \theta^2+\theta$, $v_{int} = \left(1-\theta^3-\frac{\theta^3}{1+\theta}\right)$. Notice that the equilibrium is only well defined when $\tilde{u}_{int} \leq 1$, and coincides with infinite collateral equilibrium at $\theta^2+\theta = 1$. At $(\tilde{u}_{int}, \tilde{v}_{int})$ we can evaluate

$$\begin{aligned}
&\lambda\tilde{B} = \frac{1+3\theta+\theta^2}{1+\theta}\\
&\gamma E[X(\tilde{u}_{int}, v_{int})] = v_{int}e^{-\lambda\tilde{B}} - \frac{1}{1+\theta}e^{-\lambda\tilde{B}}(1-\tilde{u}_{int})\left(1+\frac{\theta}{1+\theta}+\gamma\tilde{B}\right)\\
&= e^{-\frac{1+3\theta+\theta^2}{1+\theta}}\frac{\theta^2(2+\theta)}{1+\theta}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial\theta}e^{-\frac{1+3\theta+\theta^2}{1+\theta}}\frac{\theta^2(2+\theta)}{1+\theta} = -\frac{e^{-\frac{\theta^2}{\theta+1}-\frac{3\theta}{\theta+1}-\frac{1}{\theta+1}}\theta\left(\theta^4+2\theta^3-\theta^2-5\theta-4\right)}{(\theta+1)^3}\\
&\propto -\theta^4-2\theta^3+\theta^2+5\theta+4 > 0
\end{aligned}$$

The last inequality follows from the fact that for $\tilde{u}_{int} < 1$ we must have $\theta \leq 1$. Since this derivative is positive, the maximum expected profit (maximized over all θ) of the clearinghouse at the interior collateral equilibrium is $\theta^2+\theta = 1$. For every other θ, the clearinghouse's profit is strictly less than the infinite collateral equilibrium. It is thus never a global maximum unless it coincides with the infinite collateral equilibrium, ruling out the interior collateral equilibrium as the prevailing equilibrium.

To characterize zero collateral equilibria, we use Proposition 5. This fully characterizes the unconstrained zero collateral equilibrium (case *(iii)*).

[Step 3.] Now we rule out all constrained equilibria with complementing fees (3 cases). We start with the zero collateral equilibrium. Assume the bank is imposing complementing fees , then we must have $\tilde{G} \geq \frac{1}{2}$ by Theorem 4 (see Figures 5 and 6). In addition, since $\bar{B} = 0$, $\tilde{u} = G$. The clearinghouse's normalized expected profit is given by

$$\begin{aligned}\gamma E[X(\delta_c, C)] &= v - \frac{\theta + 2}{2(1+\theta)^2} = \frac{1}{2} - \theta\tilde{G} - \frac{\theta + 2}{2(1+\theta)^2} \leq \frac{1}{2} - \frac{\theta}{2} - \frac{\theta + 2}{2(1+\theta)^2} \\ &= -\frac{\theta^3 + \theta^2 + 1}{2(\theta+1)^2} < 0\end{aligned}$$

so the clearinghouse would not participate in clearing, hence ruling out this equilibrium.

When collateral equals infinity we must have $\xi \geq 0$ so the bank is imposing augmenting fees, ruling out the infinite collateral equilibrium.

Last we show a constrained interior collateral equilibrium with complementing fees cannot exist. Assume the contrary, since the constraint is binding and the bank is imposing complementing fees, we must have $\bar{B} = 0, v = \frac{1}{2}e^{-w} - \theta\tilde{u}$, and $\tilde{u} = \tilde{G}$ by Corollary 1 and Theorem 3. The clearinghouse's normalized expected profit is given by

$$\begin{aligned}\gamma E[X(\delta_c, C)] &= v - \frac{1}{2}e^{-w}\frac{\theta + 2}{2(1+\theta)^2} = \frac{1}{2}e^{-w} - \theta\tilde{G} - \frac{1}{2}e^{-w}\frac{\theta + 2}{2(1+\theta)^2} \\ &= \frac{1}{2}e^{-w}\frac{\theta^2 + \theta - 1}{(1+\theta)^2} - \theta\tilde{G} \leq \max\{\frac{1}{2}\frac{\theta^2 + \theta - 1}{(1+\theta)^2} - \frac{\theta}{2}, -\frac{\theta}{2}\} < 0\end{aligned}$$

This rules out constrained interior collateral equilibrium with complementing fees.

[Step 4.] Finally, we show the possible existence of constrained infinite and zero collateral equilibria with strictly augmenting fees (cases *(ii)* and *(iv)*), the possible non-existence of equilibria, whiling ruling out constrained interior collateral equilibrium with augmenting fees.

The Lagrangian associated with the clearinghouse's maximization problem is:

$$\mathcal{L}(\delta_c, C, \mu) := E[X(\delta_c, C)] + \mu\left(\delta_b e^{-\lambda\bar{B}} - G\right)$$

We assume the multiplier is positive at the critical points. In the case when it is zero, the results are the same as that given in step 2. When this "strictly complementarity" holds, the binding condition $\mu > 0$ implies, we must have that $\frac{\partial\delta_c}{\partial C}$ is the same along the level curve of the clearinghouse's expected profit and the bank's IR constraint.

Since the bank is imposing augmenting fees, along $E[X] = const.$, we have

$$\begin{aligned}e^{\lambda B}\frac{\partial E[X]}{\partial\delta_c} &= 1 + \frac{u((1+\theta)(1-u) - v)}{\theta(1-(1-u)(1+\theta))}, \\ e^{\lambda B}\frac{\partial E[X]}{\partial C} &= (1-u)\left(u + \frac{v}{1+\theta} + \frac{\theta}{(1+\theta)^2} + \frac{u(1+\theta)((1+\theta)(1-u) - v)}{\theta(1-(1-u)(1+\theta))}\right).\end{aligned}$$

Combining this with Proposition 3, we see that critical points of the Lagrangian must satisfy:

$$-\frac{(1-u)\left(u+\frac{v}{1+\theta}+\frac{\theta}{(1+\theta)^2}+\frac{u(1+\theta)((1+\theta)(1-u)-v)}{\theta(1-(1-u)(1+\theta))}\right)}{1+\frac{u((1+\theta)(1-u)-v)}{\theta(1-(1-u)(1+\theta))}}=u-1 \tag{A.35}$$

When $u=1$, we must have $v<\infty$ otherwise no clients trade. This means $C=\infty$ and we can compute the constrained infinite collateral equilbirum $(u,v,w)=(1,-\theta(\log\tilde{G}+1),\infty$, (case *(ii)*).At this point the bank's profit is zero and the clearinghouse's (normalized) expected profit is $-\theta\tilde{G}(\log\tilde{G}+1)$. Thus, for the clearinghouse to participate, $\tilde{G}\le e^{-1}$. In addition, from step 2 we see that an unconstrained infinite collateral equilibrium gives the bank's revenue $\frac{1}{\lambda}e^{-2}$, so we must have $\tilde{G}\ge e^{-2}$ for the IR constraint to be binding. We can also compute the shadow value associated with the IR constraint: $\mu=2+\log(G\lambda)$.

If $u\ne 1$, it seems that there may be a constrained "interior collateral equilibrium":

$$\frac{\left(u+\frac{v}{1+\theta}+\frac{\theta}{(1+\theta)^2}+\frac{u(1+\theta)((1+\theta)(1-u)-v)}{\theta(1-(1-u)(1+\theta))}\right)}{1+\frac{u((1+\theta)(1-u)-v)}{\theta(1-(1-u)(1+\theta))}}=1$$

or

$$\begin{aligned}
&u+\frac{v}{1+\theta}+\frac{\theta}{(1+\theta)^2}-1+\frac{u((1+\theta)(1-u)-v)}{1-(1-u)(1+\theta)}=0\\
&\frac{v}{1+\theta}+\frac{\theta}{(1+\theta)^2}-1+\frac{u(1-v)}{1-(1-u)(1+\theta)}=0\\
&v=\theta(1-u)+\frac{1}{1+\theta}
\end{aligned}$$

We will now rule out this possibility. From Corollary 1 we have:

$$\begin{aligned}
&2(1-u)=e^{-\gamma\tilde{B}-w},\\
&w=-u-\log(1-u)-\log 2-\frac{\theta^2}{1+\theta},\\
&-\gamma B=\theta\log\frac{\tilde{G}}{u},
\end{aligned}$$

so we arrive at

$$\begin{aligned}
&2(1-u)=e^{\theta\log\frac{\tilde{G}}{u}+u+\log(1-u)+\log 2+\frac{\theta^2}{1+\theta}},\\
&1=e^{\theta\log\frac{\tilde{G}}{u}+u+\frac{\theta^2}{1+\theta}}\\
&0=\log\frac{\tilde{G}}{u}+\frac{u}{\theta}+\frac{\theta}{1+\theta}
\end{aligned} \tag{A.36}$$

Again by Corollary 1 we see this imposes a restriction on $\tilde{G}$:

$$0=\log\frac{\tilde{G}}{u}+\frac{u}{\theta}+\frac{\theta}{1+\theta}\ge\log\frac{\tilde{G}}{u}+\frac{\frac{\theta}{1+\theta}}{\theta}+\frac{\theta}{1+\theta},$$

$$e^{-1} \geq \frac{\tilde{G}}{u} \geq \tilde{G}.$$

which in turn imposes a restriction on θ:

$$e^{-1} \geq \tilde{G} \geq u \geq \frac{\theta}{1+\theta}.$$

Let $\theta_H \approx 0.58$ be the solution to $e^{-1} = \frac{\theta}{1+\theta}$, we must have $\theta \leq \theta_H$.

Eq. (A.36) also imposes another restriction on θ. Since $\log u - \frac{u}{\theta}$ is maximized at $u = \theta$:

$$\frac{\theta}{1+\theta} = -\log \tilde{G} + \log u - \frac{u}{\theta} \leq 1 + \log \theta - 1 = \log \theta$$

Let $\theta_L \approx 1.93$ be the unique solution to $\log \theta = \frac{\theta}{1+\theta}$, we must have $\theta \geq \theta_L$. Altogether this means there is no θ that allows for a constrained interior collateral equilibrium with augmenting fees.

Next we turn to the constrained zero collateral equilibrium. From Corollary 1 we can write

$$(2(1-\tilde{u}))^{-\frac{1}{\theta}} = e^{\lambda \tilde{B}}$$

thus when the constraint is binding, $\tilde{u}$ must solve

$$\tilde{u}(2(1-\tilde{u}))^{\frac{1}{\theta}} = \tilde{G}.$$

Eq. (A.16) then characterizes the equilibrium (case *(*iv)). To see uniqueness of the solution, notice that $\frac{\partial u(2(1-u))^{\frac{1}{\theta}}}{\partial u} = 0$ only at $u = \frac{\theta}{1+\theta}$, so $\tilde{u}(2(1-\tilde{u}))^{\frac{1}{\theta}}$ can equal $\tilde{G}$ only at one $\tilde{u}$ on $\left[\frac{\theta}{1+\theta}, 1\right]$

Last, we see that when $\tilde{G}$ is very large, none of the equilibria presented are individually rational (case *(v)*). Since $1 \geq \tilde{u} \geq \tilde{u}e^{-\lambda B}$, the bank's normalized revenue is always bounded above by 1. A $\tilde{G}$ greater than 1 will result in the nonexistence of any feasible equilibria. Since when $\tilde{G} \leq e^{-1}$ some infinite collateral equilibrium must exist, $\tilde{G}$ must be greater than e^{-1} for this to occur.

Notice that all profit functions and IR constraints can be written in a fashion that only depends on θ and $\tilde{G}$, thus which equilibrium prevails only depends on θ and $\tilde{G}$. □

Proof of Theorem 7. This is given directly by combining Propositions 4 and 5. □

Proof of Theorem 8. We see that $\kappa(u; \theta, \alpha) = 0$ has a solution at $u = 1$ when $\alpha = 0$.

$$\frac{\partial \kappa(1; \theta, 0)}{\partial \alpha} = (1+\theta)^2 > 0.$$

By the Implicit Function theorem, there exists $\bar{\alpha}_1(\theta)$ such that we can a continuously differentiable branch of solutions $\tilde{u}^{\infty}(\alpha)$ such that

$$\tilde{u}^{\infty}(\alpha) = 1.$$

For $\theta < \frac{-1+\sqrt{5}}{2}$, we have $\theta^2 + \theta < 1$ and

$$\lim_{\alpha\to 0} \frac{\partial \kappa(u;\theta,\alpha)}{\partial u} = \theta(1-\theta^2-\theta) - \lim_{\alpha\to 0} \alpha(1+\theta)\log(2(1-u)) > 0.$$

Thus $\frac{\partial \tilde{u}^\infty(0)}{\partial \alpha} = -\frac{\frac{\partial \kappa(1;\theta,0)}{\partial \alpha}}{\frac{\partial \kappa(1;\theta,0)}{\partial u}} < 0$. By continuity of the derivative, it remains negative in a region $[0, \alpha_2(\theta)]$.

Next, from Theorem 7 we see that we can the associated fee and collateral by plugging in $u = u^\infty(\alpha)$:

$$\tilde{v}(u,\alpha) = \frac{\alpha - \theta^2(1+\alpha) + ((1+\theta)^2 + (-1+\theta+\theta^2))u - (1+\theta)u^2 - \alpha(1-u)\log 2(1-u)}{\alpha+u}$$

$$\tilde{w}(u,\alpha) = \frac{\theta^2 - 2\theta u - \theta^2 u - u\log 2(1-u)}{\alpha+u}$$

Since we see

$$\begin{aligned}
&\lim_{\alpha\to 0, u\to 1} \frac{\partial \tilde{v}(u,\alpha)}{\partial \alpha} \\
&= \lim_{\alpha\to 0, u\to 1} \frac{\theta^2u^2 - 2\theta^2 u + \theta^2 + 2\theta u^2 - 2\theta u + u^2\log(2(1-u)) - u\log(2(1-u))}{(\alpha+u)^2} = 0. \\
&\lim_{\alpha\to 0, u\to 1} \frac{\partial \tilde{v}(u,\alpha)}{\partial u} \\
&= \lim_{\alpha\to 0, u\to 1} \frac{\alpha^2\theta^2 + \alpha^2\theta + 2\alpha\theta^2 - 2\alpha\theta u + 2\alpha\theta - \alpha u + \theta^2 - \theta u^2 - u^2}{(\alpha+u)^2} \\
&+ \lim_{\alpha\to 0, u\to 1} \frac{\alpha^2\log(2(1-u)) + \alpha\log 2(1-u)}{(\alpha+u)^2} \\
&= -1 - \theta + \theta^2 + \lim_{\alpha\to 0, u\to 1} \frac{\alpha^2\log(2(1-u)) + \alpha\log 2(1-u)}{(\alpha+u)^2} < 0.
\end{aligned}$$

So there is a region $[0, \alpha_3(\theta)]$, in which $\frac{d\tilde{v}}{d\alpha} = \frac{\partial \tilde{v}}{\partial \alpha} + \frac{\partial \tilde{v}}{\partial u}\frac{\partial \tilde{u}^\infty}{\partial \alpha} > 0$ Similarly, we can write:

$$\begin{aligned}
&\lim_{\alpha\to 0, u\to 1} \frac{\partial \tilde{w}(u,\alpha)}{\partial u} \\
&= \lim_{\alpha\to 0, u\to 1} \frac{\alpha\theta^2 u - \alpha\theta^2 + 2\alpha\theta u - 2\alpha\theta + \alpha u - \alpha\log(2(1-u)) + \alpha u\log(2(1-u)) + \theta^2 u - \theta^2 + u^2}{(1-u)(\alpha+u)^2} \\
&= \lim_{\alpha\to 0, u\to 1} \frac{-\alpha(1-u)\log(1-u) + 1}{(1-u)(\alpha+u)^2} \geq 0.
\end{aligned}$$

Since $w \geq 0$ at $\alpha = 0$, obviously $\lim_{\alpha\to 0, u\to 1} \frac{\partial \tilde{w}(u,\alpha)}{\partial \alpha} < 0$. Together this means there is a region $[0, \alpha_4(\theta)]$, in which $\frac{d\tilde{w}}{d\alpha} = \frac{\partial \tilde{w}}{\partial \alpha} + \frac{\partial \tilde{w}}{\partial u}\frac{\partial \tilde{u}^\infty}{\partial \alpha} < 0$. To finish the proof, take $\tilde{\alpha}(\theta) = \min_{i=1,2,3,4} \alpha_i(\theta)$. □

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